A LEFSCHETZ FIXED POINT FORMULA FOR ELLIPTIC DIFFERENTIAL OPERATORS

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Communicated by E. Spanier, August 20, 1965

Introduction. The classical Lefschetz fixed point formula expresses, under suitable circumstances, the number of fixed points of a continuous map \( f : X \to X \) in terms of the transformation induced by \( f \) on the cohomology of \( X \). If \( X \) is not just a topological space but has some further structure, and if this structure is preserved by \( f \), one would expect to be able to refine the Lefschetz formula and to say more about the nature of the fixed points. The purpose of this note is to present such a refinement (Theorem 1) when \( X \) is a compact differentiable manifold endowed with an elliptic differential operator (or more generally an elliptic complex). Taking essentially the classical operators of complex and Riemannian geometry we obtain a number of important special cases (Theorems 2, 3). The first of these was conjectured to us by Shimura and was proved by Eichler for dimension one.

1. The main theorem. Let \( X \) be a smooth compact manifold and let \( E, F \) be smooth complex vector bundles over \( X \). A differential operator from \( E \) to \( F \) means a linear map \( d : \Gamma(E) \to \Gamma(F) \) on the spaces of smooth sections which is given in local coordinates by a matrix of partial differential operators with smooth coefficients. By an elliptic complex \( E \) on \( X \) we mean a sequence \( E_0, E_1, \ldots, E_n \) of smooth vector bundles over \( X \) and a sequence of differential operators \( d_i : \Gamma(E_i) \to \Gamma(E_{i+1}) \) so that

(i) \( d_{i+1}d_i = 0 \) for \( i = 1, \ldots, n-1 \), and

(ii) the sequence of symbols

\[
\cdots \to E_{i,x} \xrightarrow{\sigma_i(x, \xi)} E_{i+1,x} \to \cdots
\]

is exact for all \( x \in X \) and all nonzero cotangent vectors \( \xi \) to \( X \) at \( x \). Here \( \sigma_i \) denotes the symbol of \( d_i \), and \( d_i \) as well as \( \sigma_i \) is formally put equal to zero for \( i < 0 \) and \( i \geq n \).

In view of (i) the cohomology groups \( H^i(E) = \text{Kernel } d_i / \text{Image } d_{i-1} \), are well defined and it is a consequence of (ii) that \( H^i(E) \) is finite-dimensional. Note that, if \( n = 1 \), we are dealing with a single elliptic operator \( d_0 \). \( H^0 \) is then the space of solutions of \( d_0 u = 0 \) and \( H^1 \) can

\[1\] This work was supported in part by the National Science Foundation.