THE GEOMETRY OF G-STRUCTURES[1]

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1. **Introduction.** Differential geometry studies differentiable manifolds and geometric objects or structures on them. It is now customary to distinguish it from differential topology by the presence of a structure in addition to the differentiable structure. What a differential geometric structure is or should be is a matter of taste. At the present state of the field a definition general enough to include all the significant structures will certainly contain many uninteresting ones. Among the attempts at a general definition is the notion of a geometric object initiated by Oswald Veblen [**130**].

Not all the geometrical structures are "equal". It would seem that the riemannian and complex structures, with their contacts with other fields of mathematics and with their richness in results, should occupy a central position in differential geometry. A unifying idea is the notion of a G-structure, which is the modern version of a local equivalence problem first emphasized and exploited in its various special cases by Elie Cartan [**36**], [**41**], [**129**]. Generally we will restrict ourselves in this article to the discussion of problems which fall under the notion of a G-structure.

Two general problems are of importance:

I. *Existence or nonexistence of certain structures on a manifold.*

EXAMPLE 1. A positive definite riemannian structure always exists.

EXAMPLE 2. On a compact manifold M a nowhere zero differentiable vector field exists if and only if the Euler-Poincaré characteristic of M is zero.

EXAMPLE 3. One may ask whether a nonzero vector field exists on M (supposed to be compact and orientable) which is parallel with respect to a riemannian metric. By Hodge's harmonic forms a necessary condition is that the first Betti number b^1 of M is $\geqq 1$. One can further prove that the second Betti number $b^2 \geqq b^1 - 1$ (cf. §5). These conditions are probably not sufficient.

II. *Local and global properties of a given structure.*

EXAMPLE 1. For a riemannian structure this means riemannian geometry.

EXAMPLE 2. If we are only interested in the existence or nonexist-

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