WIENER AND INTEGRATION IN FUNCTION SPACES

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1. Evolution of Mathematics is, by and large, a continuous process and its growth and progress seldom deviate greatly from the natural historical lines. It is because of this that we tend, in retrospect, to admire most those developments which though born well outside it have grown to join and to enrich the mainstream of our science.

It was the great fortune and the great achievement of Norbert Wiener to initiate such a development when, in the early twenties, he introduced a measure, now justly bearing his name, in the space of continuous functions.

2. Let us first review briefly some of the background.

At about 1905, almost simultaneously, and quite independently of each other (in fact, using wholly different approaches) A. Einstein and M. Smoluchowski provided a theory of the peculiar erratic motion of small particles suspended in liquids first described in 1828 by the English botanist Brown.

The theory can be summarized as follows:

(a) For simplicity one confines one's attention to the displacement of the Brownian particle in some chosen direction and one can thus speak of the one-dimensional Brownian motion.

(b) The motion is Markoffian and homogeneous in time; i.e. the probability of finding the particle at times $t_1, t_2, \cdots, t_n$ $(0<t_1<t_2<\cdots<t_n)$ in the intervals $(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)$ is given by the formula

$$(2.1)\quad \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\cdots\int_{\alpha_n}^{\beta_n} P(x_0\,|\,x_1;t_1)P(x_1\,|\,x_2;t_2-t_1)\cdots \cdots P(x_{n-1}\,|\,x_n;t_n-t_{n-1})dx_1\cdots dx_n$$

where $P(x|y;t)$ is the probability density of finding the particle at y at time t if it started at x at $t=0$.

(c) For $\Delta t\to 0$ one has asymptotically

$$\langle\Delta x\rangle = \int_{-\infty}^{\infty}(x-x_0)P(x_0\,|\,x;\Delta t)\,dx \sim F(x_0)\Delta t,$$

$$\langle(\Delta x)^2\rangle = \int_{-\infty}^{\infty}(x-x_0)^2P(x_0\,|\,x;\Delta t)\,dx \sim 2D\Delta t,$$