ON THE OSCILLATIONS AND LEBESGUE CLASSES OF A FUNCTION AND ITS POTENTIALS[1]

BY R. K. JUBERG

Communicated by A. Zygmund, September 7, 1965

Suppose $f \in L^r(R)$, $r \geqq 1$, R a cube in E^n. Then one knows from Sobolev's theorems [5] that the potential

$$(0.1) \qquad P \to \int_R f(Q) \mid P - Q \mid^{-\alpha n} dQ, \qquad 0 < \alpha < 1,$$

is in $L^\sigma(R)$, $\sigma^{-1} > \alpha - 1 + r^{-1}$, where $|P - Q|$ denotes the Euclidean distance between P, $Q \in E^n$.

In this note we demonstrate a certain converse proposition. For a non-negative function $f \in L^r(R)$, $r \geqq 1$, we assume the potential (0.1) to be in $L^s(R)$, $0 \leqq s^{-1} < \alpha - 1 + r^{-1}$ (s a positive real number or ∞), and in addition make an assumption on the "oscillations" of f (cf. §1). Then we can conclude that f is summable to powers exceeding r.

We express the so-called "oscillatory" conditions and present the main theorem, Theorem A, in the next section. The proof of the theorem is direct and simple. In §2 we state a parallel theorem, Theorem B, wherein the assumption on the potential is replaced by the hypothesis that the function is in some "Morrey class" (cf. Morrey [3]; or also Campanato [1]). Theorem B is described perhaps more accurately as a corollary to the proof of Theorem A. In the last section, §3, we show how these results can be indirectly deduced. Therein we use a lemma from a paper by Semenov [4] which relates "Marcinkiewicz classes" (cf. e.g., Zygmund [6]) with "Lorentz" spaces. The conclusion follows then from the inclusion relations between Lorentz spaces and Lebesgue spaces (cf. Lorentz [2]).

1. **The principal result.** Let f be a non-negative function summable over R, a cube in E^n. For S any measurable set in E^n we indicate its (Lebesgue) measure by meas S. Set

$$(1.1) \qquad E(x) = \{P: P \in R, \ f(P) > x\}.$$

CONDITION I. For some $a > 0$, $0 \leqq \lambda \leqq 1$ (a may depend on λ)

[1] This work was supported in part by the Office of Naval Research under Contract No. Nonr-710(54).