ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR VOLTERRA EQUATIONS

BY R. K. MILLER

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In this note we show how certain known results for delay differential equations can be extended to systems of integral equations of the form

$$(1) \qquad x(t) = f(t) + \int_0^t a(t-s)g(s, x(s))\,ds \qquad (t \geqq 0).$$

We make the following assumptions:

(A1) $f(t)$ is uniformly continuous and bounded on $0 \leqq t < \infty$,

(A2) $a(t)$ is a square matrix whose entries are $L_1(0, \infty)$,

(A3) $g(t, x)$ is continuous in (t, x) for $0 \leqq t < \infty$, $|x| < \infty$ and g is uniformly almost periodic in t uniformly on compact subsets of x in real n-space R^n, and

(A4) $x(t)$ is a bounded solution of (1) for $0 \leqq t < \infty$.

Let Ω be the positive limit set of $x(t)$. We refer to [2] for the definitions and properties of almost periodic functions and limit sets. The analog for integral equations of [2, Theorem 1] is

THEOREM 1. *If* (A1)–(A4) *are satisfied, then to each point z in Ω there corresponds a sequence $t_m \to \infty$ as $m \to \infty$ and functions $G(t, x)$, $X(t)$ and $F(t)$ such that*

(i) $\text{limit}_{m\to\infty} |x(t+t_m) - X(t)| + |f(t+t_m) - F(t)| = 0$ *uniformly on compact subsets of* $-\infty < t < \infty$,

(ii) $\text{limit}_{m\to\infty}\, g(t+t_m, x) = G(t, x)$ *uniformly for all t and for x on compact sets, and*

(iii) *on the interval* $-\infty < t < \infty$, $X(t) \in \Omega$ *and*

$$(2) \qquad X(t) = F(t) + \int_{-\infty}^t a(t-s)G(s, X(s))\,ds.$$

PROOF. As is well known in harmonic analysis the convolution of an L_1 function with an essentially bounded function yields a uniformly continuous function. Hence $x(t)$ is bounded and uniformly continuous on the interval $0 \leqq t < \infty$.

Given z in Ω let $\{t_m\}$ be a sequence such that $t_m \to \infty$ and $x(t_m) \to z$ as $m \to \infty$. Define $x_m(t) = x(t+t_m)$ and $f_m(t) = f(t+t_m)$ for $t \geqq -t_m$. Since

$$x_m(t) = f_m(t) + \int_{-t_m}^t a(t-s)g(s+t_m, x_m(s))\,ds,$$