CLASSIFICATION OF MARKOV CHAINS WITH A GENERAL STATE SPACE

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1. **Introduction.** Let X be a general abstract space of points x, and $\mathfrak{X}$ a Borel σ-field of sets in X. Let us consider a *transition function* $p(\cdot, \cdot)$ of the arguments $x \in X$, $A \in \mathfrak{X}$ (see [2, p. 190]) which may be, however, sub-stochastic, i.e. where the usual assumption $p(x, X) = 1$ is replaced by $p(x, X) \leqq 1$. The iterates $p^{(n)}$ of p are defined as usual (see e.g. [2, p. 191]).

We shall always suppose that p is *irreducible*, i.e. that the measures $\nu_x = \sum_{n=1}^{\infty} 2^{-n} p^{(n)}(x, \cdot)$ are equivalent for all $x \in X$. A measure μ is called sub-invariant if it is σ-finite, not identically zero, and if

$$\int_X p(x, A)\mu(dx) \leqq \mu(A) \qquad \text{for all } A \in \mathfrak{X}. \tag{1}$$

If in (1) the sign of equality holds for all $A \in \mathfrak{X}$, then μ is called invariant.

THEOREM 1. *If $\mathfrak{X}$ is generated by a denumerable class of sets, then there always exists a sub-invariant measure for any p.*

The proof follows by a simple application of the results in [5] and [8] whenever $\sum_{n=1}^{\infty} p^{(n)}(x, A) = \infty$ for each x and each A satisfying $\nu_x(A) > 0$, and by putting $\mu = \sum_{n=1}^{\infty} p^{(n)}(x_0, \cdot)$ whenever $\sum_{n=1}^{\infty} p^{(n)}(x_0, A) < \infty$ for some x_0 and some A such that $\nu_x(A) > 0$. However, there have been given also other, more complicated, conditions for the existence of a sub-invariant measure (see [8], [4]).

Let us assume in the sequel that we have some sub-invariant measure μ, and that this μ is equivalent to each ν_x. It may be seen that the latter assumption causes no loss of generality (see [8]).

Define the operator T_α, $1 \leqq \alpha \leqq \infty$ (see [8]), in the space $L_\alpha(\mu)$ by

$$T_\alpha f = \int_X f(y) p(\cdot, dy). \tag{2}$$

2. **Classification of transition functions.** Our basic classification is given by the following

THEOREM 2. *Each irreducible transition function p having a sub-invariant measure μ belongs precisely to one of the following types: either $\sum_{n=1}^{\infty} p^{(n)}(x, A) = \infty$ for each A such that $\mu(A) > 0$ and each x (p is*