THE TOPOLOGICAL COMPLEMENTATION PROBLEM

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Let Σ be the lattice of all topologies definable on an arbitrary set E. Then Σ is a complete lattice with the trivial topology, $\{\emptyset, E\}$, as the least element and the discrete topology, $P(E)$, as the greatest element.

The problem of complementation in the lattice Σ has been outstanding for some time although several investigators have provided partial solutions. Hartmanis [6] first showed that Σ was a complemented lattice if the set E was finite and Gaifman [4] proved Σ was complemented if E was countable. Berri [1], using the results of Gaifman, was able to provide complements for certain special topologies such as a topological group with a dense, nonopen, countable subgroup.

It is the purpose of this paper to introduce the lattice of principal topologies, and to establish that the lattice Σ of all topologies on a set E is complemented.

The following theorems are stated without proof. The full details will be published elsewhere.

1. **Principal topologies.** A topology $\tau \in \Sigma$ is called an ultraspace if the only topology finer than τ is the discrete topology. Fröhlich [3] shows that every topology τ is the infimum of ultraspaces finer than τ. For a filter $\mathfrak{F}$ on E and a point $x \in E$, Fröhlich defined $\mathfrak{S}(x, \mathfrak{F})$ to be the family of sets $P(E - \{x\}) \cup \mathfrak{F}$, which is a topology. He proved the ultraspaces are the topologies of the form $\mathfrak{S}(x, \mathfrak{U})$ where $x \in E$ and $\mathfrak{U}$ is an ultrafilter on E different from the principal ultrafilter at x, $\mathfrak{U}(x)$. The set of ultraspaces may be divided into two classes each of which generates a sublattice of Σ. One of these sublattices consists of all T_1-topologies. The other is called the lattice of principal topologies.

Every topology τ on E is the infimum of all ultraspaces on E finer than τ. If also $\tau = \inf \{\mathfrak{S}(x, \mathfrak{U}(y)) \mid \mathfrak{S}(x, \mathfrak{U}(y)) \geqq \tau\}$ then τ will be called a principal topology.

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