A DENSITY THEOREM FOR LACUNARY FOURIER SERIES

BY J. W. MOELLER AND P. O. FREDERICKSON

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I. Introduction. Let Λ be a nonempty subset of the integers, and let $L^2(\Lambda)$ denote the closed subspace of $L^2(0, 2\pi)$ spanned by the exponentials $\{e^{i\lambda x} | \lambda \in \Lambda\}$. Suppose we are given the values of an arbitrary function f in $L^2(\Lambda)$ on a fixed interval of positive length δ. When can we determine the values of f *outside* that interval? A precise answer to this question will be announced below, after some essential terminology has been introduced to help us handle the problem.

Accordingly, let χ_δ denote the indicator function for the interval $(0, \delta)$, and let $A_\delta(f) = \chi_\delta f$; in words, $A_\delta(f)$ is simply the function which coincides with f on the interval $(0, \delta)$ but vanishes elsewhere. We say that a set of integers Λ is an *extrapolation set of length* ρ if the mapping A_δ: $L^2(\Lambda) \to \chi_\delta L^2(\Lambda)$ has a bounded inverse for $\delta > \rho$ but fails to have a bounded inverse whenever $\delta < \rho$. It is easy to see that every set of integers has a unique extrapolation length ρ, and A_δ^{-1} will extrapolate functions in $L^2(\Lambda)$ from $(0, \delta)$ onto $(0, 2\pi)$ as long as $\delta > \rho$. Of course, since $L^2(\Lambda)$ is translation invariant, there is nothing sacred about our choice of the interval $(0, \delta)$; any other interval of length δ would serve the same purpose.

It turns out that the extrapolation length of a prescribed set can be explicitly computed if we know how sparsely the points in this set are distributed. The appropriate concept to use in this connection is the notion of *uniform outer density*. Following Kahane [2], we define the uniform outer density of a set Λ to be

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \left\{ \sup_{-\infty < \sigma < \infty} N(\sigma, \alpha) \right\},$$

where $N(\sigma, \alpha)$ represents the number of points of Λ contained in the interval $[\sigma, \sigma + \alpha)$. Our main result expresses the exact relationship between outer density and extrapolation length.

THEOREM. *Let Λ be a set of integers whose uniform outer density is $D(\Lambda)$. Then Λ is an extrapolation set of length ρ if and only if $\rho = 2\pi D(\Lambda)$.*

A detailed proof of this Theorem, further generalized to include L^2 spaces of exponential functions with gaps in their spectra, will be published elsewhere [7]. In what follows we briefly outline our plan of attack to expose the main ideas.