SUBSPACES OF $C(S)_\beta$, THE SPACE (l^∞, β), AND (H^∞, β)

BY JOHN B. CONWAY[1]

Communicated by R. C. Buck, August 16, 1965

The author has shown [2] that if S is a paracompact locally compact space then every β-weak * countably compact subset of $M(S)$ is β-equicontinuous (see [2] for definitions and notation). If we define a *strong Mackey space* to be a topological vector space E such that every weak * compact (not necessarily convex and circled) subset of E^* is equicontinuous, then $C(S)$ with the strict topology β is a strong Mackey space whenever S is paracompact. A natural problem is to characterize those subspaces of $C(S)$ which are (strong) Mackey spaces if they have the relative strict topology and if $C(S)_\beta$ is a (strong) Mackey space. In particular, we may ask this question for a paracompact space S.

Along these lines it is known that the completion of a Mackey space is a Mackey space, but the converse is false. In fact, $C(S)_\beta$ is the completion of $C_0(S)_\beta$, but $C_0(S)_\beta$ is never a Mackey space (unless S is compact), since the norm topology is always stronger than β and yields the same adjoint $M(S)$.

At present we have no solution to our question, but we can give an answer in the case where S is the space of positive integers. Also, we show that H^∞, the space of bounded analytic functions on the open unit disk D, is not a Mackey space if it has the relative β topology, even though $C(D)_\beta$ is a strong Mackey space.

The difficulties encountered in attacking the general problem may be visualized as follows. Let E be a subspace of $C(S)$ and i: $E_\beta \rightarrow C(S)_\beta$ the injection map, with i^*: $M(S) \rightarrow E_\beta^*$ its adjoint. In order to show that a subset $H \subset E_\beta^*$ is β-equicontinuous it is necessary and sufficient to show that there is a β-equicontinuous subset $H_1 \subset M(S)$ such that $i^* H_1 = H$. Therefore, if $C(S)_\beta$ is a Mackey space and $H \subset E_\beta^*$ is β-weak * compact convex and circled, then to show that H is β-equicontinuous we must find a β-weak * compact convex circled set $H_1 \subset M(S)$ such that $i^* H_1 = H$. Since E_β^* with its β-weak* topology is topologically isomorphic to a quotient space of $M(S)$, it would seem that what is needed is a version of a theorem of Bartle and Graves [4, p. 375] where both domain and range have their β-weak* topolo-

[1] These results are taken from the author's dissertation, written while he held a National Science Foundation Cooperative Fellowship at Louisiana State University. Partial support was also furnished by NSF Grant GP 1449. The author wishes to express his appreciation to Professor H. S. Collins for his advice and encouragement.