EXPONENTIATION OF OPERATOR LIE ALGEBRAS ON BANACH SPACES[1]

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Communicated by S. Smale, April 7, 1965

1. **Introduction.** By a *C^∞ Lie algebra of operators on a Banach space* $\mathfrak{X}$ we shall mean a pair $(g, \mathfrak{D})$ consisting of (a) a dense linear subset ("domain") $\mathfrak{D}$ of the space $\mathfrak{X}$ and (b) a finite dimensional real vector space g of operators $X, Y, Z \cdots$ defined on $\mathfrak{D}$ such that $X\mathfrak{D} \subset \mathfrak{D}$ for all $X \in g$ ("C^∞ condition") and such that the *commutator* Lie product $[X, Y] = XY - YX$ carries $g \times g$ into g.

It is well known that every strongly continuous representation U of a Lie group G on $\mathfrak{X}$ gives rise to a number of different domains $\mathfrak{D}$ of C^∞ vectors for U on which the Lie algebra g of G may be represented to give such a C^∞ Lie algebra $(g, \mathfrak{D})$ (cf. Segal [10], Gårding [4], Harish-Chandra [5], Cartier and Dixmier [2] and Nelson [8]). Here $U(G)$ is a generalized exponential of $(g, \mathfrak{D})$. Therefore we will call a C^∞ Lie algebra of operators *exponentiable* in case the simply connected Lie group G whose Lie algebra is isomorphic with g has a strongly continuous representation U on $\mathfrak{X}$ such that when $f \in \mathfrak{D}$:

$$\lim_{t \to 0} t^{-1}[U(\exp tX)f - f] = Xf \tag{1}$$

(here we have identified $(g, \mathfrak{D})$ with the Lie algebra of G). We will discuss the question: When is a C^∞ Lie algebra of operators on $\mathfrak{D}$ exponentiable? Nelson [8] gives a sufficient condition for the case of a Lie algebra of skew-symmetric operators on a Hilbert space H.

An operator X will be called a *pregenerator on* $\mathfrak{D}$ in case it has a closure $\overline{X}$ generating a strongly continuous one parameter group of operators in the sense of Hille and Phillips [6] (denoted by $U[t, X]$ here). Sufficient conditions for this are given in [6]. A counter-example of Nelson [8] refutes the natural conjecture that every C^∞ Lie algebra of pregenerators (each individually exponentiable) is exponentiable in the sense discussed above. We give a number of different sufficient conditions for the exponentiability of C^∞ Lie algebras of pregenerators on Banach spaces. Various of the results can be extended to suitably defined "C^1" and "C^2" Lie algebras.

[1] This work was supported in part by the National Science Foundation, both through a Fellowship and through NSF GT 1611. Several of the results obtained form a part of a dissertation submitted to Princeton University in candidacy for the degree of Ph.D. The author extends his thanks to Professor E. Nelson for his many suggestions and encouragement.