ORLICZ SPACES AND NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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Nonlinear elliptic differential equations of order m acting in a space of m dimensions often occupy a special position in more general theories. In this paper we shall study one aspect of this situation. The nonlinear problem under consideration will be the variational approach to eigenvalue problems for nonlinear elliptic partial differential equations as developed by the author in [1], [2], [3], N. Levinson [8] and F. E. Browder [4]. We shall study nonlinearities with exponential type growth, thus filling a gap in the earlier work. A special result applicable in this context, is a theorem of F. John and L. Nirenberg [6], which allows an extension of the Sobolev Imbedding Theorem to Orlicz spaces. The author extends hearty thanks to Professors R. Juberg, W. Littman and N. Meyers for helpful suggestions with this work. This research was partially supported by NSF-GP 2280.

I. **Imbedding Sobolev spaces in Orlicz spaces.** Let G be a bounded domain in N-dimensional real Euclidean space R^N. We consider various classes of real-valued functions defined on G and their integrals with respect to N-dimensional Lebesgue measure. The Sobolev space $W_{m,p}(G)$ consists of all function $u(x)$ in $L_p(G)$, whose derivatives (in the sense of L. Schwartz) up to order m are also in $L_p(G)$. $\mathcal{W}_{m,p}(G)$ is the closure in $W_{m,p}(G)$ of $C_0^\infty(G)$. $\mathcal{W}_p^\infty(G)$ is a Banach space with respect to the norm

$$\|u\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{0,p}^p.$$

Suppose $\phi(t)$ is a real-valued, continuous, convex even function of a real variable such that $\lim_{t\to 0} \phi(t)/t = 0$ and $\lim_{t\to\infty} \phi(t)/t = \infty$. Then the Orlicz class $L_\phi(G)$ with respect to the function $\phi(t)$ consists of all functions $u(x)$ such that $\int_G \phi(u(x))dx < \infty$. An Orlicz class can be made into a Banach space $L_{\phi^*}(G)$ by associating with $\phi(t)$ its complementary function $\theta(t)$ and defining

$$\|u\| = \sup_v \int_G uv\, dx, \quad \text{where} \quad \int_G \theta(v)\, dx \leq 1.$$