APPROXIMATION OF BOUNDED FUNCTIONS
BY CONTINUOUS FUNCTIONS

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We shall show that every bounded function on a paracompact space has a best approximation by continuous functions, and characterize the functions whose best approximators are unique. This is a special case of a measure-theoretic problem, whose setting is as follows. Let X be a topological space and μ a Borel measure on X which assigns positive mass to each nonempty open set, and has the property that $\mu(Y)=0$ if Y intersects a neighborhood of each point in a μ-null set. The latter condition is automatically fulfilled if each open cover of X has a countable subcover. Let L^∞ be the space of essentially bounded real-valued μ-measurable functions on X, and give it the semi-norm $\|f\|=$essential sup $|f|$. The bounded continuous functions on X form a closed subspace C of L^∞. We say that $g\in C$ is a *best approximator* to $f\in L^\infty$ if $\|f-g\|=\text{dist}(f, C)=\inf\{\|f-h\|: h\in C\}$.

If $f\in L^\infty$ and $x\in X$, $f^*(x)=\lim\ \sup_{y\to x} f(y)=\inf\{$ess sup of f over U: U is a neighborhood of $x\}$; $f_*=\lim\ \inf_{y\to x} f(y)$ has a similar definition. It is easy to verify that the functions f^* and f_* are defined everywhere, and are upper semi-continuous (usc) and lower semi-continuous (lsc) respectively.

PROPOSITION. *If X is any topological space and $f\in L^\infty$, then* $2\ \text{dist}(f, C)\geqq d(f)\equiv\sup\{f^*(y)-f_*(y): y\in X\}$.

PROOF. If $f^*(x)-f_*(x)>d(f)-\epsilon$ and $g\in C$ then one or the other of $\lim\ \sup_{y\to x}(f(y)-g(y))$ and $\lim\ \sup_{y\to x}(g(y)-f(y))$ is greater than $\frac{1}{2}(d(f)-\epsilon)$.

THEOREM 1. *If X is paracompact, then $g\in C$ is a best approximator to $f\in L^\infty$ if, and only if, $f^*-\frac{1}{2}d(f)\leqq g\leqq f_*+\frac{1}{2}d(f)$; every $f\in L^\infty$ has such a best approximator; and* $\text{dist}(f, C)=1/2d(f)$.

PROOF. Since $f_*+\frac{1}{2}d(f)\geqq f^*-\frac{1}{2}d(f)$, the first pair of inequalities is equivalent to the condition that for every $\epsilon>0$ and every $x\in X$, there be a neighborhood U of x such that (ess sup$|f(y)-g(y)|: y\in U$) $\leqq\frac{1}{2}d(f)+\epsilon$. This in turn is equivalent to the assertion that for every $\epsilon>0$, $|f(y)-g(y)|>\frac{1}{2}d(f)+\epsilon$ only on a μ-null set, which says that $\|f-g\|\leqq\frac{1}{2}d(f)$. It remains only to show that there is a continuous function which satisfies these inequalities. Since $f^*-\frac{1}{2}d(f)$