THE COHOMOLOGY OF CLASSIFYING SPACES OF H-SPACES

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Let G denote an associative H-space with unit (e.g. a topological group). We will show that the relations between G and a classifying space B_G are more readily displayed using a geometric analog of the resolutions of homological algebra. The analogy is quite sharp, the stages of the resolution, whose base is B_G, determine a filtration of B_G. The resulting spectral sequence for cohomology is independent of the choice of the resolution, it converges to $H^*(B_G)$, and its E_2-term is $\mathrm{Ext}_{H(G)}(R, R)$ (R = ground ring). We thus obtain spectral sequences of the Eilenberg-Moore type [5] in a simpler and more geometric manner.

1. **Geometric resolutions.** We shall restrict ourselves to the category of compactly generated spaces. Such a space is Hausdorff and each subset which meets every compact set in a closed set is itself closed (a k-space in the terminology of Kelley [3, p. 230]). Subspaces are usually required to be closed, and to be deformation retracts of neighborhoods.

Let G be an associative H-space with unit e. A right G-action on a space X will be a continuous map $X \times G \to X$ with $xe = x$, $x(g_1g_2) = (xg_1)g_2$ for all $x \in X$, $g_1, g_2 \in G$. A space X with a right G-action will be called a G-space. A G-space X and a sequence of G-invariant closed subspaces $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ such that $X_0 \neq \emptyset$, $X = \bigcup_{i=0}^{\infty} X_i$, and X has the weak topology induced by $\{X_i\}$ will be called a *filtered* G-space.

1.1. DEFINITION. (a) A filtered G-space X is called *acyclic* if for some point $x_0 \in X_0$, X_n is contractible to x_0 in X_{n+1} for every n.

(b) A filtered G-space X is called *free* if, for each n, there exists a closed subspace D_n ($X_{n-1} \subset D_n \subset X_n$) such that the action mapping $(D_n, X_{n-1}) \times G \to (X_n, X_{n-1})$ is a relative homeomorphism.

(c) A filtered G-space X is called a *G-resolution* if X is both free and acyclic.

Under the restrictions we have imposed on subspaces, the acyclicity condition implies that X is contractible.

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