NONLINEAR MONOTONE OPERATORS AND CONVEX SETS IN BANACH SPACES

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Introduction. Let X be a real Banach space, X^* its conjugate space, (w, u) the pairing between w in X^* and u in X. If C is a closed convex subset of X, a mapping T of C into X^* is said to be monotone if

$$(Tu - Tv, u - v) \geqq 0 \tag{1}$$

for all u and v in C.

It is the object of the present note to prove the following theorem:

THEOREM 1. *Let C be a closed convex subset of the reflexive Banach space X with $0 \in C$, T a monotone mapping of C into X^*. Suppose that T is continuous from line segments in C to the weak topology of X^* while $(Tu, u)/\|u\| \to +\infty$ as $\|u\| \to +\infty$.*

Then for each given element w_0 of X^*, there exists u_0 in C such that

$$(Tu_0 - w_0, u_0 - v) \leqq 0 \tag{2}$$

for all v in C.

If $C = X$, Theorem 1 asserts that $Tu_0 = w_0$ and reduces to a theorem on monotone operators proved independently by the writer [1] and G. J. Minty [9] and applied to nonlinear elliptic boundary value problems by the writer in [2], [3], and [6]. (See also Leray and Lions [7].) If $C = V$, a closed subspace of X, the conclusion of Theorem 1 is that $Tu_0 - w_0 \in V^{\perp}$, which yields a variant of the generalized form of the Beurling-Livingston theorem proved by the writer in [4] and [5]. The conclusion of Theorem 1 for $C = X$ was extended by the writer to classes of densely defined operators (see [6] for references) and in [5] to multivalued mappings.

It is easily shown that Theorem 1 generalizes and includes as a special case the following linear theorem of Stampacchia, which has been applied by the latter to the proof of the existence of capacitary potentials with respect to second-order linear elliptic equations with discontinuous coefficients:[2]

THEOREM 2. *Let H be a real Hilbert space, C a closed convex subset of H, $a(u, v)$ a bilinear form on H which is separately continuous in u*

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[2] C. R. Acad. Sci. Paris **258** (1964), 4413–4416.

Added in proof. A result similar to Theorem 1 has recently been obtained jointly by Hartman and Stampacchia (in an as yet unpublished paper) who also give a very interesting application to existence theorems for second order nonlinear elliptic equations.