WAVE PROPAGATION NEAR A SMOOTH CAUSTIC[1]

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A caustic is an envelope of a family of rays of geometrical optics. At a caustic, the usual equations of geometrical optics are not valid. We give a formal asymptotic series, valid both near and away from a smooth caustic, which satisfies the differential equation exactly. In the case of high-frequency oscillations, the solution is represented in terms of the Airy function and its derivative. There is a more general formulation (corresponding to a progressing wave expansion) in which the solution is expanded in terms of solutions of the Tricomi equation. Our procedure can be applied to general linear hyperbolic or time-reduced partial differential equations. The coefficients in our expansion are nearly the same as the coefficients which appear in ordinary geometrical optics; the only essential difference is that a factor which is singular at the caustic has been removed. Our terminology is explained in R. Courant, *Methods of mathematical physics*, Vol. II, Ch. VI.

In order to illustrate our procedure, we consider the reduced wave equation $\Delta u+k^2u=0$, and we give only the leading term in the expansion. All of the essential features are illustrated in this special problem. We write the first term as

$$(1)\quad u(x) = \exp(ik\theta(x))\left[A(-k^{2/3}\rho(x))g(x) + \frac{i}{k^{1/3}}A'(-k^{2/3}\rho(x))h(x)\right].$$

Here A denotes the Airy function; we have

$$(2)\qquad A''(t) = tA(t),$$

and

$$(3)\qquad A'''(t) = tA' + A(t).$$

The functions θ, ρ, g and h are determined below. The caustic will be obtained by setting $\rho(x)=0$. Applying the differential operator, using (2) and (3) and collecting terms, we obtain

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