PROJECTIVE METRICS IN DYNAMIC PROGRAMMING

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It has been shown by Birkhoff [2], [3] that Hilbert's projective metric [4] may be applied to a variety of problems involving linear mappings of a function space into itself. In this note we shall point out that essentially the same metric may be applied to some nonlinear mappings which frequently arise in dynamic programming [1].

Let X be some set, and let P denote the set of all nonnegative real-valued functions which have domain X and are not identically zero. We define an extended real-valued function θ on $P\times P$ as follows:

$$\theta(f, g) = \log\left[\left(\sup_{x\in X}\frac{f(x)}{g(x)}\right)\cdot\left(\sup_{x\in X}\frac{g(x)}{f(x)}\right)\right].$$

In computing the ratios, we take $0|0$ to be 1, and $a|0$ to be ∞ if $a\neq 0$. It is easy to show that θ is an extended pseudo-metric on P. $\theta(f, g)=0$ implies that $f=\lambda g$ for some constant $\lambda>0$. We say that a subset P^* of P is "metric" if θ is an extended metric on P^*. That is, if for any $f, g\in P^*$, $\theta(f, g)=0$ if and only if $f=g$.

Let L be a map of P into P. If

$$\sup_{x\in X}\frac{Lf(x)}{Lg(x)} < \sup_{x\in X}\frac{f(x)}{g(x)} \quad \text{for all } f, g\in P$$

such that $0<\theta(f, g)<\infty$ then we say L is "ratio reducing on P." Note that if L is ratio reducing on P it follows at once that $\theta(Lf, Lg)<\theta(f, g)$ for all $f, g\in P$ such that $0<\theta(f, g)<\infty$.

Thus L is a contraction mapping with respect to the pseudo-metric θ. Similar definitions apply on any subset of P. Many linear transformations have been shown [2], [3] to be ratio reducing (or at least ratio nonincreasing). A family $\{L_\lambda\}$ (λ ranging over some set of parameters Λ) is said to be "uniformly ratio reducing" if, given f, g,

$$\sup_{x\in X}\frac{L_\lambda(f(x))}{L_\lambda(g(x))} \leqq \sup_{x\in X}\frac{f(x)}{g(x)} - \delta_{f,g} \quad \text{for all } \lambda\in\Lambda,$$

where $\delta_{f,g}>0$ may depend on f and g but does *not* depend on λ. Note that if Λ is a finite set then the family $\{L_\lambda\}$ is uniformly ratio reducing if each of its members is ratio reducing.

THEOREM. *If the family* $\{L_\lambda : \lambda\in\Lambda\}$ *is uniformly ratio reducing,*