ESSENTIALLY POSITIVE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

BY GARRETT BIRKHOFF AND LEON KOTIN

Communicated by F. Browder, May 17, 1965

Consider an essentially positive [2] system of linear DE's, that is, any system of the form

$$(1) \qquad dx_i/dt = \sum q_{ij}(t)x_j, \qquad q_{ij}(t) > 0 \text{ if } i \neq j.$$

This maps the positive hyperoctant C of real $(x_1, \cdots, x_n)$-space into itself. The effect of (1) in a small increment of time dt is, using formulas of Ostrowski [3], to map C into an *interior* polyhedral cone whose projective diameter is given for $P = \exp[Q(t)dt] = I + Q(t)dt + \cdots$ by

$$(2) \qquad \ln\{\sup_{i,j,k,l}(p_{ki}p_{lj}/p_{kj}p_{li})\}.$$

This is maximized asymptotically (as $dt \downarrow 0$) by setting $k=i$, $l=j$, so that the numerator approaches 1. Hence the projective diameter of $e^{Q(t)dt}(C)$ is, asymptotically,

$$(3) \qquad \Delta = -\ln\{\inf_{i\neq j}[(q_{ij}q_{ji})dt^2]\}, \qquad q_{ij} = q_{ij}(t).$$

By a basic theorem of [1], all projective distances in C are therefore contracted by a factor at most

$$(4) \qquad \tanh(\Delta/4) = (1 - e^{-\Delta/2})/(1 + e^{-\Delta/2}) = 1 - \psi(t)dt, \quad \text{where } \psi(t) = 2[\inf_{i\neq j} q_{ij}(t)q_{ji}(t)]^{1/2}.$$

This proves the following basic result.

LEMMA. *For any essentially positive system* (1) *of linear* DE'*s, all projective distances in C are contracted by an asymptotic factor at most $1-\psi(t)dt$ in the time interval $(t, t+dt)$, where $\psi(t)$ is given by* (4).

Integrating with respect to t, we deduce the

THEOREM. *For any essentially positive system* (1) *of linear* DE'*s, let $\theta(\mathbf{x}(t), y(t))$ denote the projective distance in C between two solutions of* (1) *which are positive on* $[0, \infty)$. *Then*

$$(5) \qquad \theta(\mathbf{x}(t), y(t)) \leqq \theta(\mathbf{x}(0), y(0)) \exp\left[-\int_0^t \psi(s)ds\right], \qquad t > 0.$$

For example, consider the interesting case $d^2x/dt^2 = p(t)x$, $p(t) > 0$. Then