DECISION METHODS IN THE THEORY OF ORDINALS[1]

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Communicated by D. Scott, May 21, 1965

For an ordinal α, let RS(α), the restricted second order theory of $[\alpha, <]$, be the interpreted formalism containing the first order theory of $[\alpha, <]$ and quantification on monadic predicate variables, ranging over all subsets of α. For a cardinal γ, RS(α, γ) is like RS(α), except that the predicate variables are now restricted to range over subsets of α of cardinality less than γ. $\omega = \omega_0$ and ω_1 denote the first two infinite cardinals. In this note I will outline results concerning RS(α, ω_0), which were obtained in the Spring of 1964 (detailed proofs will appear in [8]), and the corresponding stronger results about RS(α, ω_1), which were obtained in the Fall of 1964.

The binary expansion of natural numbers can be extended to ordinals. If $x < 2^\alpha$, let ϕx be the finite subset $\{u_1, \cdots, u_n\}$ of α, given by $x = 2^{u_1} + \cdots + 2^{u_n}$, $u_n < \cdots < u_1$. ϕ is a one-to-one map of 2α onto all finite subsets of α. Let Exy stand for $(\exists u)[x = 2^u \wedge u \in \phi y]$, and note that the algorithm $i+j=s$, for addition in binary notation can be expressed in RS(α, ω_0). It now is easy to see that the first order theory FT$[2^\alpha, +, E]$ is equivalent to RS(α, ω_0), in the strong sense that the two theories merely differ in the choice of primitive notions; the binary expansion ϕ yields the translation. Similarly, RS(α, γ) can be reinterpreted as a first order theory. We will state our results in one of the two forms, and leave it to the reader to translate.

THEOREM 1. *For any α, there is a decision method for truth of sentences in $RS(\alpha, \omega_0)$. The same sentences are true in $RS(\alpha, \omega_0)$ and $RS(\beta, \omega_0)$, if and only if, $\alpha = \beta < \omega^\omega$ or else $\alpha, \beta \geqq \omega^\omega$ and have the same ω-tail.*

If $\alpha = z + \omega^y + \omega^n c_n + \cdots + \omega^0 c_0$, $y \geqq \omega$, then $z + \omega^y$ is called the *ω-head* of α, and $\omega^n c_n + \cdots + \omega^0 c_0$ is called the ω-tail of α.

THEOREM 2. *For any ordinals $\beta > \alpha > \omega^\omega$, $[2^\beta, +, E]$ is an elementary extension of $[2^\alpha, +, E]$, if and only if, α and β have the same ω-tail. The elementary embedding is then given by $h(2^{\alpha_0}x + y) = 2^{\beta_0}x + y$, whereby $x < 2^\tau$, $y < 2^{\alpha_0}$, τ is the common ω-tail of α and β, α_0 and β_0 are respectively the ω-heads of α and β.*

[1] This work was supported in part by grant GP-2754 from the National Science Foundation.