THE ISOPERIMETRIC INEQUALITY FOR MULTIPLY-CONNECTED MINIMAL SURFACES[1]

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Many proofs have been given of the isoperimetric inequality for minimal surfaces of the type of the disc, which was discovered by T. Carleman [2] in 1921. The question, however, to find a similar inequality for minimal surfaces of higher topological type seems never to have been attacked in the literature. On the basis of new results [3], [4] such an estimate can be derived for multiply-connected minimal surfaces of planar type; and we want to state it here, and sketch the proof, for the case of a doubly-connected minimal surface, answering in part problems 25 and 26 formulated in [5]:

Let S be a minimal surface of the type of the circular annulus of area A (finite or infinite), bounded by two distinct Jordan curves Γ_1 and Γ_2 of lengths L_1 and L_2, respectively (finite or infinite). If these curves are rectifiable, then the area of S is finite, and the inequality $(L_1+L_2)^2 - 4A > 0$ is satisfied.

The numerical value of the constant 4 can easily be improved. But the question for the best value of this constant—which undoubtedly is 4π—must be left open.

Consider a minimal surface $S = \{\mathfrak{x} = \mathfrak{x}(u, v);\ (u, v) \in \overline{P}\}$, where $\overline{P}$ is the closure of the ring domain $P = \{u, v;\ 0 < r_1^2 < u^2+v^2 < r_2^2 < \infty\}$. The vector $\mathfrak{x}(u, v) \in C^2(P) \cap C^0(\overline{P})$ satisfies in P the regularity condition $|\mathfrak{x}_u \times \mathfrak{x}_v| > 0$, the condition of vanishing mean curvature $H = 0$, and maps the bounding circles of P onto the curves Γ_1 and Γ_2 in a monotonic manner.

The minimal surface has a conformal representation, i.e. a representation where, in addition to having the above properties, the vector $\mathfrak{x}(u, v)$ satisfies in P the relations $\mathfrak{x}_u^2 = \mathfrak{x}_v^2$, $\mathfrak{x}_u \cdot \mathfrak{x}_v = 0$, and maps the bounding circles of P topologically onto Γ_1 and Γ_2. We set $w = u + iv = \rho e^{i\theta}$, and we shall use interchangeably the notations $\mathfrak{x}(u, v)$ and $\mathfrak{x}(\rho, \theta)$. Once the surface is given in a conformal representation the regularity condition $\mathfrak{x}_u^2 > 0$ is of no consequence.

For $r_1 < r < r_2$ let $\gamma(r)$ be the circle $\{u, v;\ u^2+v^2 = r^2\}$, $\Gamma(r)$ its image on S, and $L(r)$ the length of $\Gamma(r)$. Applying a device due to L. Bieberbach [1] and T. Radó [6] it is seen that $L(r) \leqq \mathrm{Max}(L_1, L_2)$.

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