HERGLOTZ TRANSFORMATION AND H^p THEORY

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Considerable work has been done in recent years to generalize a certain part of the theory of holomorphic functions on the unit disc, related to the Hardy classes H^p, as well as to determine a proper setting for the resulting abstract H^p theory. The reader is referred to the excellent accounts in Hoffman [5] and Hoffman-Rossi [6]; see also Lumer [7] and Srinivasan-Wang [9].

In this note we intend to describe how an approach to the above theory at its most general level, based on what we call the Herglotz transformation,[2] leads to: (i) advantages in deriving the theory; (ii) stronger and more explicit forms of known results; (iii) new results, which we believe shed light on the nature of the subject and yield applicable information. The Herglotz transformation corresponds in the classical case to the integral transformation defined by what is sometimes called the Herglotz kernel. In §1 we introduce the Herglotz transformation under most general conditions, and show that if A is any subalgebra of $L^\infty(m)$ containing 1, m a positive measure multiplicative on A, then a certain form of logmodularity will hold even if m is not unique as a representing measure on A. From this one can, in particular, settle a question implicitly left open by Hoffman-Rossi [6]: whether to simply assume density of Re A in $L^p_R(m)$ for all p finite will yield the usual H^p theory and imply the uniqueness of m in the sense of [6] (it is shown in [6] that density of Re A in L^1_R or L^2_R will not suffice). The answer is affirmative. In §2 we extend the transformation under uniqueness assumption on m, and use this in §3 to derive the full H^p theory. Once the properties of the transformation are known, many results will follow very directly without need of treating separately H^p spaces for different values of p. The crucial fact that Re A is dense in L^p_R for all p finite (under uniqueness assumption on m) is proved here without appeal to some nontrivial tool foreign to H^p theory (such as Lemma 6.6, [5]). As an example of "more precise forms of results," we prove a Szegö theorem giving not

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[2] Related ideas were used recently, in connection with special questions, by Devinatz [3], [2], dealing with Dirichlet algebras, and by Lumer [8], in a setting without uniform approximation assumptions.