THE GENUS OF K_n, $n=12(2^m)$

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Communicated by G. Whaples, February 25, 1965

Introduction. It is the object of this note to display a minimal imbedding of K_n, the complete graph on n vertices with $n=12(2^m)$, $m=0, 1, 2, \cdots$. (See [5] for notation and terminology.) A minimal imbedding of K_n with $n=12(2^m)(2t+1)$ and $t=1, 2, \cdots$ has also been obtained, but will be discussed elsewhere. All the imbeddings are triangular, and hence an easy computation involving the Euler formula shows that the genus of K_n is $(n-3)(n-4)/12$ for $n=12s$, $s=1, 2, 3, \cdots$. For the connection between these results and the Heawood conjecture one may consult Ringel [3].

The method employed was announced by Gustin [2] and generalized by him to include the case in which the group used to name the vertices is non-Abelian, and "knobs" are permitted in the "quotient" network. The technique involves the delicate matching of a group to the geometry of a network, and in all other applications the group has been Abelian. On the other hand it can be shown that if the index of the solution is to be 1, as it is here, then the use of non-Abelian groups is essential.

The group. An appropriate group will be defined as the normal (or semi-direct) product (see [1, p. 88]) of a certain finite group and a group of its automorphisms. The finite group will be the *additive* group in a certain finite field and the *multiplicative* structure of the field will play a leading role in defining the automorphisms.

The following facts about finite fields will be used. (See [4, pp. 91–118].)

(1) For $k=1, 2, \cdots$ there is a finite field $\mathrm{GF}(2^k)$, the Galois field of order 2^k.

(2) If $p \in \mathrm{GF}(2^k)$ then $p+p=0$.

(3) The multiplicative group in $\mathrm{GF}(2^k)$, here called $F^*(2^k)$, is cyclic. Suppose θ is a generator, then the order of θ is (2^k-1).

(4) If $F^+(2^k)$ is the additive group in $\mathrm{GF}(2^k)$, then $1, \theta, \theta^2, \cdots, \theta^{k-1}$ is a basis for $F^+(2^k)$ over $F^+(2)$.

Using the exponential notation for an automorphism, define an automophism α of $F^+(2^k)$ by the linear extension of the following mapping of the generators: