GENERALIZED UNITARY OPERATORS[1]

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1. Let C be the complex field and Γ be the unit circle $\{\lambda \in C: |\lambda| = 1\}$. For a non-negative integer m or for $m = \infty$, let $C^m(\Gamma)$ be the space of all m-times continuously differentiable functions on Γ. (Here we consider Γ as a C^∞-manifold in the natural way. Thus, any $f \in C^m(\Gamma)$ can be identified with an m-times continuously differentiable periodic function $f(\theta)$ of a real variable θ with period 2π.) $C^m(\Gamma)$ is an algebra as well as a Banach space if m is finite, a Fréchet space if $m = \infty$, with the usual sup-norms for derivatives.

We shall say that a mapping γ of Γ into C is a *C^m-curve* if γ can be extended onto a neighborhood V of Γ (the extended map will also be denoted by γ) in such a way that it is one-to-one on V and γ and γ^{-1} are both m-times continuously differentiable (as functions in two variables) on V and $\gamma(V)$ respectively.

Let E be a Hausdorff locally convex space over C such that the space $\mathcal{L}(E)$ of all continuous linear operators on E endowed with the bounded convergence topology is quasi-complete.

2. **$C^m(\gamma)$-operators.**

DEFINITION. Let γ be a C^m-curve. $T \in \mathcal{L}(E)$ is called a *$C^m(\gamma)$-operator* if there exists a continuous algebra homomorphism W of $C^m(\Gamma)$ into $\mathcal{L}(E)$ such that $W(1) = I$ and $W(\gamma) = T$. If γ is the identity map: $\gamma(\theta) = e^{i\theta}$, then a $C^m(\gamma)$-operator is called a *C^m-unitary operator*. (Cf. Kantrovitz' approach in [1].)

THEOREM 1. *If T is a $C^m(\gamma)$-operator, then* $\mathrm{Sp}(T) \subseteq \gamma(\Gamma)$.[2]

If H is a Hilbert space, $T \in \mathcal{L}(H)$ is a C^0-unitary operator if and only if it is similar to a unitary operator on H. In this sense, C^m-unitary operators on E generalize the notion of unitary operators on a Hilbert space.

The homomorphism W in the above definition is uniquely determined by T and γ. Thus, we call W the $C^m(\gamma)$-representation for T. The uniqueness can be derived from the following approximation theorem: *Given a C^m-curve γ, let λ_0 be a point inside the Jordan curve*

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[2] $\mathrm{Sp}(T)$ is the spectrum of T in Waelbroeck's sense. See [2] for the definition.