GEOMETRICAL REALIZATION OF ISOMORPHISMS BETWEEN PLANE GROUPS

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Communicated by L. Bers, February 5, 1965

Let G denote the group of all isometries of the hyperbolic plane. If we take the Poincaré model, the subset of the complex plane with $\mathcal{I}z>0$, and the metric $|dz|/\mathcal{I}z$, then G consists of orientation-preserving maps

$$w = \frac{az+b}{cz+d}, \qquad a, b, c, d \text{ real } ad - bc = 1$$

and orientation-reversing maps

$$w = \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad a, b, c, d \text{ real } ad - bc = -1.$$

We denote the upper half-plane by D, so that $[G, D]$ is a transformation group with the topology on G and D induced by the parameters a, b, c, d, z.

If H is a discrete subgroup of G, we call the transformation group $[H, D]$ a *non-euclidean crystallographic group* (NEC group). The structure of those NEC groups which consist entirely of orientation-preserving maps (Fuchsian groups) has been well-known for some time, but the structure of general NEC groups has only recently been obtained [3]. The aim of this note is to prove the following

THEOREM. *Let $[H, D]$ be an NEC group with compact orbit-space D/H. Let $\alpha: H\rightarrow H$ be an automorphism of the group H. Then the automorphism α can be geometrically realized, i.e., there is a homeomorphism $t: D\rightarrow D$ such that, for all $h\in H$, if h^α denotes the image of h under the automorphism α, we have*

$$t(hz) = h^\alpha t(z).$$

The NEC groups include, as special cases, the fundamental groups of all compact surfaces except the sphere, projective plane, torus and Klein bottle. For such groups, the result is due to Nielsen, and can be re-expressed in the form:

Every isomorphism between the fundamental groups of two compact surfaces can be induced by a homeomorphism of one surface on the other.

For Fuchsian groups, the result has recently been proved by Zieschang [4], using combinatorial methods.