DIMENSION OF METRIC SPACES AND HILBERT'S PROBLEM 13[1]

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In 1957 A. N. Kolmogorov [1] and V. I. Arnol'd [2] obtained the following result (answering Hilbert's conjecture in the negative):

THEOREM. *For every integer* $n \geqq 2$ *there exist continuous real functions* ψ^{pq}, *for* $p=1, 2, \cdots, n$ *and* $q=1, 2, \cdots, 2n+1$, *defined on the unit interval* $E^1=[0, 1]$, *such that every continuous real function* f, *defined on the* n*-dimensional unit cube* E^n, *is representable in the form*

$$f(x_1, \cdots, x_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^{n} \psi^{pq}(x_p) \right],$$

where the functions χ_q *are real and continuous.*

The proof of the theorem relies on two properties of E^1, namely, E^1 is compact and of dimension 1. (By dimension we shall always mean covering dimension.) This paper generalizes the work of Kolmogorov and Arnol'd to obtain the following result:

THEOREM 2. *For* $p=1, 2, \cdots, m$ *let* X^p *be a compact metric space of finite dimension* dp, *and let* $n=\sum_{p=1}^{n} d_p$. *There exist continuous functions* $\psi^{pq}: X^p \rightarrow [0, 1]$, *for* $p=1, \cdots, m$ *and* $q=1, 2, \cdots, 2n+1$, *such that every continuous real function* f *defined on* $\prod_{p=1}^{m} X^p$ *is representable in the form*

$$f(x_1, \cdots, x_m) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^{m} \psi^{pq}(x_p) \right],$$

where the functions χ_q *are real and continuous.*

The proof of Theorem 2 makes use of the following new characterization of dimension of metric spaces which is of interest in itself.

THEOREM 1. *A metric space* X *is of dimension* $\leqq n$ *if and only if for each open cover* $\mathcal{C}$ *of* X *and each integer* $k \geqq n+1$ *there exist* k *discrete families of open sets* $\mathcal{U}_1, \cdots, \mathcal{U}_k$ *such that the union of any* $n+1$ *of the* $\mathcal{U}_i$ *is a cover of* X *which refines* $\mathcal{C}$.

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