A NONTOPOLOGICAL 1-1 MAPPING ONTO E^3

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It is well known that any 1-1 mapping from one compact Hausdorff space onto another is a homeomorphism. It is also an easy consequence of the Brouwer theorem on invariance of domain that any 1-1 mapping from one Euclidean space onto another is topological. Recently considerable interest has been expressed by several authors in the degree to which the rather stringent requirement that the domain space be Euclidean can be relaxed in the latter case. This paper gives an example which strongly indicates that very little if any relaxation can be allowed.

We will exhibit a 1-1 mapping which is not a homeomorphism from a connected, locally connected and locally compact metric space onto an open polyhedral 3-cell embedded in E^3. This is contrary to a result supposedly proven by V. V. Proizvolov[1] asserting that any 1-1 mapping from a connected paracompact space onto E^n is a homeomorphism. The domain space of this example is simply a polyhedral subset of E^3 on which the mapping is piecewise linear.

The domain space A of our mapping is an open cube with four vertical triangular columns rising out of the top, with their bases symmetrically about a square. The hypotenuse of each triangle covers half of one side of the square with one of its ends at a vertex of the square. The remainder of the triangle and its interior lie outside the square. Note that neither the square nor the line segments bounding the base triangles of the columns are in our space. In fact the only piece of the boundary that is in the space is one collar around each column starting part way up the column and extending up to but not including the top. The line segments bounding this collar are also not in the space (see Figure 1).

This space may be described analytically as

$$\begin{aligned}&\{-4 < z < 0,\ -2 < x < 2,\ -2 < y < 2\}\\ &\quad \cup \bigcup \big[\{t_2 > 1, t_2 - t_1 - 1 < 0, t_2 + t_1 - 2 < 0, 0 \leqq z \leqq 2^{1/2}\}\\ &\quad \cup \{t_2 \geqq 1,\ t_2 - t_1 - 1 \leqq 0,\ t_2 + t_1 - 2 \leqq 0,\ 2^{1/2} < z < 1 + 2^{1/2}\}\big],\\ &\qquad\qquad (t_1, t_2) \in \{(x, y), (-x, -y), (y, -x), (-y, x)\}.\end{aligned}$$

[1] Dokl. Akad. Nauk SSSR **151** (1963), 1286–1287; English transl., Soviet Math. Dokl. **4** (1963), 1194.