CO-IMMUNE RETRACEABLE SETS

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Our notation and terminology basically follows that found in [2], except with regard to notation for unions and intersections; in a few instances we cite other references for special terms.

The following two propositions are established by a fairly straightforward moveable-markers technique; either proof is only a minor variation on the other.[1]

PROPOSITION A. *Let α be an infinite recursively enumerable set. Then there is a countably infinite collection Γ of retraceable sets γ_i such that* (i) $i \neq j \Rightarrow \gamma_i \cap \gamma_j = \emptyset$, (ii) *each γ_i is the unique infinite set retraced by a certain basic general recursive retracing function (for the notions of retracing function and basic retracing function, see* [5]), (iii) $\alpha = \bigcup \Gamma$, *and* (iv) $\alpha - \gamma_i$ *is immune for all i.*

PROPOSITION B. *Let α be an infinite recursively enumerable set, and τ an infinite recursive subset of α such that $\alpha - \tau$ is also infinite. Then there is a recursive function f such that, for each i, $f(i)$ indexes a basic, general recursive retracing function which retraces a unique infinite set γ_i, where* (i) $i \neq j \Rightarrow \gamma_i \cap \gamma_j = \emptyset$, (ii) *each γ_i has exactly one number in common with $\alpha' \cup \tau$, and* (iii) $(\alpha - \tau) - \gamma_i$ *is immune for all i.*

It was shown by Yates, in [5] (in answer to a question of Dekker and Myhill), that there are basic retracing functions, some of them retracing *unique* infinite sets, which do not retrace any infinite *recursive* set. In each of Yates' examples, all of the sets retraced by such functions have nonimmune complements. The above propositions demonstrate the existence of examples in which an infinite set α is retraced by a basic function and α has immune complement. In any example of this latter type, the function in question *must* retrace a *unique* infinite set, which, of course, cannot be recursive.

We remark that all of the sets γ_i obtained by us in proving Propositions A and B are, owing to the nature of the proofs, *hyperimmune* (for the notion of *hyperimmunity*, see, e.g., [5]). This is closely related to the following general assertion:

[1] We are indebted to Paul Young for a conversation which took place in August, 1963. At that time he made a suggestion which has proved to be susceptible of elaboration into proofs of Propositions A and B.