# THE HAUPTVERMUTUNG AND THE POLYHEDRAL SCHOENFLIES THEOREM

BY P. M. RICE

Communicated by M. L. Curtis, December 28, 1964

1. **Introduction.** M. L. Curtis [1] has conjectured that the double suspension of a Poincaré manifold is a 5-sphere. If this is true, it gives counterexamples to the Hauptvermutung, the closed star conjecture, and the polyhedral Schoenflies theorems. We prove here that the only way to get a noncombinatorial triangulation of a manifold is, essentially, to multiply suspend a combinatorial manifold which is not a sphere. As a corollary, we establish that, modulo the Poincaré conjecture, one of the polyhedral Schoenflies theorems is equivalent to the Hauptvermutung.

2. **Terminology.** The Hauptvermutung is the conjecture that any two triangulations of an $n$-manifold are piecewise linearly homeomorphic. It is convenient to consider two conjectures which together imply the Hauptvermutung. The first is that any triangulation of an $n$-manifold is combinatorial (meaning that the link of any vertex is a combinatorial $(n-1)$-sphere), and the second is that any two combinatorial triangulations of an $n$-manifold are piecewise linearly homeomorphic. We will call the first of these $H(n)$. $H(n)$ is known for $n=1, 2, 3$. PS$(n)$ will denote the conjecture that, if a combinatorial $(n-1)$-sphere $S$ is embedded as a subcomplex of a triangulated $n$-sphere $T$, then $S$ is locally flat in $T$. PS$(n)$ is known for $n=1, 2, 3$. $P(n)$ will be the $n$-dimensional Poincaré conjecture, which is known except for $n=3, 4$. $S^n$ will be any space homeomorphic to the $n$-sphere, $X \cong Y$ means $X$ is homeomorphic to $Y$, $X \circ Y$ is the topological join of $X$ and $Y$, and $S(X)$ is the suspension of $X$.

3. **Main result.**

THEOREM. *If there is a noncombinatorial triangulation of an $n$-manifold $M$, then there is a combinatorial $m$-manifold $K^m$, $m \geqq 3$, such that*

(i) *$K^m$ is a homology $m$-sphere but $K^m \not\cong S^m$ and*

(ii) $K^m \circ S^{n-m-1} \cong S^n$.

PROOF. Let $v$ be a vertex of $M$ such that LK$(v, M)$, the link of $v$ in $M$, is not a combinatorial $(n-1)$-sphere. If LK$(v, M) = K^{n-1}$ is a combinatorial manifold, then $S(K^{n-1}) \cong S^n$ by Theorem 4 of [2] and the theorem is proved. By induction, if $K^p \circ S^{n-p-1} \cong S^n$ but $K^p$ is not