ALGEBRAIC INTEGRATION THEORY

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1. Introduction. Although the theory of the abstract Lebesgue integral has strong intrinsic architectural articulation, the very importance of the theory for a variety of mathematical applications has tended to cloud its basic simplicity and elegance. The large number of approaches and formalisms, each justified by its appropriateness in connection with particular applications, has given the theory as a whole an appearance of heterogeneous complexity which is in fact significantly specious and a hindrance to a deeper understanding of the theory and to its further application. While it would probably be widely granted that integration serves as a focal point for real analysis, the focus has been more virtual than real; the ultimate convergence of the multitude of generally similar but technically distinct theories has been felt rather than proved.

On the other hand, the coherence and algebraic simplicity of the basic theory has been visible and virtually taken for granted for a long time by those working in abstract analysis. The replacement of the sigma-ring of all measurable sets by the abstract Boolean ring of measurable sets modulo null sets is natural from a fundamental statistical viewpoint, and provides probably the earliest approach displaying such features. This Boolean ring approach, which has now been elaborated by many publications, most notably those of Carathéodory and his associates, is, however, an inconvenient one for most applications, inasmuch as these deal mainly with functions, which enter into the Boolean ring approach only in a rather circumlocutory fashion. Around the later thirties there appeared, as a fruit of the spectral and representation theory originating in the well-known work of von Neumann, Stone, and Gelfand, simple abstract characterizations of measure-theoretic function spaces, as ring and/or lattices. However, no comprehensive development of integration theory along such lines took place at that time.

It was the development of a variety of new theories, rather than the desire to embellish old ones, which primarily has led to the development of a complex of results, methods, and ideas here somewhat loosely referred to as 'algebraic integration theory.' The introduction of a new term such as this requires some explanation and justifica-

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