FUNCTION SPACES[1]

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1. **Introduction.** What I shall say is directed towards the explicit description and study of individual functionals and operators. I first consider the function spaces $C_n^m(D)$, B, K, Z (defined below) and their adjoints. Then I consider the factorization of operators.

If X is a normed linear space, its *adjoint*, or conjugate, or dual, X^*, is defined as the space of linear continuous functionals on X, with norm

$$\|F\| = \sup_{x\in X;\|x\|=1} |Fx|, \qquad F \in X^*.$$

The space X^* is determined by X. For some X, our knowledge of X^* is complete and useful. This is the case if X is a Hilbert space, or an L^p-space, $p \geqq 1$, or the space $C_0(D)$ of continuous functions on a compact domain D [2, Chapter 4]. For some X, as we shall see, our knowledge of X^* is incomplete.

Definitive theorems about the spaces $C_n(I)^*$, B^*, K^*, and Z^* are given in §2 and §4. These theorems provide accessible standard forms for Fx, $x\in X$, and explicit procedures for calculating $\|F\|$, where $F\in X^*$ and X is $C_n(I)$, B, or Z. Theorem 6 provides an accessible form, free of Stieltjes integrals, for Fx, $x\in B$, where $F\in K^*$.

The theorems of §3 about $C_n^m(D)^*$ appear to be new. Theorem 2 asserts the existence of a standard form for Fx, $x\in C_n^m(D)$, where $F\in C_n^m(D)^*$. Theorems 3 and 4 describe the functional 0 as an element of $C_1^m(I)^*$ and $C_2^2(I)^*$.

Just as X determines X^*, so a pair X, Y of normed linear spaces determines the space $\mathfrak{I}(X, Y)$ of linear continuous operators on X to Y. If we wish to study an operator $T_0\in\mathfrak{I}(X, Y)$, the properties of T_0 common to all elements of $\mathfrak{I}(X, Y)$ may be insufficient to provide an accessible form for T_0x, $x\in X$. It is often useful to study T_0 as an individual and, if possible, to write T_0 as a product of linear continuous operators. Such factorizations and their use in the theory of approximation are considered in §5.

Theorem 10 is a dual of Fubini's theorem.

2. **The space** $C_n(I)$. Let I be a compact linear interval and n a nonnegative integer. The space $C_n(I)$ consists of functions on I which

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