# MORSE THEORY FOR $G$-MANIFOLDS

BY ARTHUR WASSERMAN

Communicated by L. Zippin, October 20, 1964

Morse theory relates the topology of a Hilbert manifold [3, §9], $M$, to the behavior of a $C^\infty$ function $f: M \to \mathbf{R}$ having only nondegenerate critical points. In applying Morse theory to the study of $G$-manifolds, i.e., manifolds with a compact Lie group $G$ acting as a differentiable transformation group, one must, of course, use maps in the category, i.e., equivariant maps. However, if $x$ is a critical point of an equivariant function then $gx$ is also a critical point for any $g \in G$, hence one must allow critical orbits or, more generally, critical submanifolds.

In §1 we give the necessary definitions and notation. In §2 we extend the results of R. Palais in [3] to study an invariant $C^\infty$ function $f: M \to \mathbf{R}$ on a complete Riemannian $G$-space $M$, where in addition to $f$ satisfying condition (C) [3, §10], we require that the critical locus of $f$ be a union of nondegenerate critical manifolds in the sense of Bott [1]. In §3 we show that if $M$ is finite-dimensional then any invariant $C^\infty$ function on $M$ can be $C^k$ approximated by a $C^\infty$ invariant function whose critical orbits are nondegenerate. Together with the results of §2 this provides an analogue for $G$-manifolds of the Smale handlebody decomposition technique. Proofs will be given elsewhere.

1. **Notation and definition.** $G$ will denote a compact Lie group and $M$ a $C^\infty$ Hilbert manifold. If $\psi: G \times M \to M$ is the differentiable action of $G$ on $M$, then, for any $g \in G$, $\bar{g}: M \to M$ will denote the map given by $\bar{g}(m) = \psi(g, m)$; $\psi(g, m)$ will also be shortened to $gm$. If $M$, $N$ are $G$-manifolds, then $f: M \to N$, is equivariant if $f \circ \bar{g} = \bar{g} \circ f$ for all $g \in G$; $f$ is invariant if $f \circ \bar{g} = f$ for all $g \in G$. The tangent bundle $T(M)$ of a $G$-manifold $M$ is a $G$-manifold with the action $gX = d\bar{g}_p(X)$, for $X \in T(M)_p$. If $E$ and $B$ are $G$-manifolds and $\pi: E \to B$ is a Hilbert vector bundle [2], then $\pi$ is said to be a $G$-vector bundle if, for each $g \in G$, $\bar{g}: E \to E$ is a bundle map. Note that $\pi$ is then equivariant as is the zero-section. If, in addition, $\pi$ has a Riemannian metric, $\langle\ ,\ \rangle$, and each $g \in G$ acts isometrically, then $\pi$ will be called a Riemannian $G$-vector bundle. $M$ will be called a Riemannian $G$-space if $T(M) \to M$ is a Riemannian $G$-vector bundle. Let $f: M \to \mathbf{R}$ be an invariant $C^\infty$ function. The gradient vector field, $\nabla f$, on $M$, is defined by $\langle \nabla f, X\rangle = df_p(X)$ for $X \in T(M)_p$ and, since $f$ is invariant, $g\nabla f_p, \langle X\rangle = \langle \nabla f_p, g^{-1}X\rangle = df_p(g^{-1}X) = d(f \circ \bar{g}^{-1})_{gp}(X) = df_{gp}(X) = \langle \nabla f_{gp}, X\rangle$ for all $X \in T(M)_{gp}$