INVARIANCE OF THE ESSENTIAL SPECTRUM[1]

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Let A be a densely defined linear operator in a Banach space X. Wolf [4] defines the *essential spectrum* of A as the complement of Φ_A, the set of those complex λ for which $A-\lambda$ is closed and

(a) $\alpha(A-\lambda)$, the multiplicity of λ, is finite,

(b) $R(A-\lambda)$, the range of $A-\lambda$, is closed,

(c) $\beta(A-\lambda)$, the codimension of $R(A-\lambda)$, is finite.

This set, which we denote by $\sigma_{ew}(A)$, has the desirable property of being invariant under arbitrary compact perturbations. The largest subset of the spectrum having this property will be denoted by $\sigma_{em}(A)$. It is obtained by adding to $\sigma_{ew}(A)$ those points of Φ_A for which $i(A-\lambda)\equiv\alpha(A-\lambda)-\beta(A-\lambda)\neq 0$.

Browder [2] has given a different definition. It is equivalent to adding to $\sigma_{em}(A)$ those spectral points which are not isolated. We denote the essential spectrum according to this definition by $\sigma_{eb}(A)$. It has the advantage of excluding only isolated points of the spectrum, but is more delicate with respect to perturbations.

In this paper we give sufficient conditions for the invariance of each of the sets $\sigma_{ew}(A)$, $\sigma_{em}(A)$, $\sigma_{eb}(A)$ under perturbations.

Let B be a linear operator in X with $D(A)\subseteq D(B)$. It is called A-bounded if $\|Bx\|\leqq\text{const.}(\|x\|+\|Ax\|)$ for all $x\in D(A)$. It is called A-compact if $\|x_n\|+\|Ax_n\|\leqq\text{const.}$ for $\{x_n\}\subseteq D(A)$ implies that $\{Bx_n\}$ has a convergent subsequence. We shall call B *A-closed* if $x_n\to x$, $Ax_n\to y$, $Bx_n\to z$ implies that $x\in D(B)$ and $Bx=z$. It will be called *A-closable* if $x_n\to 0$, $Ax_n\to 0$, $Bx_n\to z$ implies $z=0$.

THEOREM 1. *If B is A-compact, then*

$$\sigma_{ew}(A+B)\subseteq\sigma_{ew}(A), \tag{1}$$

$$\sigma_{em}(A+B)\subseteq\sigma_{em}(A). \tag{2}$$

If, in addition, one of the following holds

(a) *A is closable,*

(b) *B is $(A+B)$-closable,*

(c) *B is A-closed,*

then

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