THE GAUSS-BONNET THEOREM AND THE TAMAGAWA NUMBER

BY TAKASHI ONO

Communicated by G. D. Mostow, September 14, 1964

This note is an outline of some of the author's recent work on the Tamagawa numbers. Our purpose is to express the Tamagawa number of Chevalley groups in terms of Euler-Poincaré characteristics, Bernoulli numbers and Gaussian curvatures.

The method can, of course, be applied to derive other formulas of the same type. We treat, however, only Chevalley groups, because it is in the present case that the main ideas of our method are most clear. The details and further results will be given elsewhere.

Let G_0 be a connected semisimple real Lie group without compact factors and center, $\mathfrak{g}_0$ its Lie algebra and put $\mathfrak{g}=\mathfrak{g}_0\otimes \boldsymbol{C}$. Let K be a maximal compact subgroup of G_0 and let $\mathfrak{k}$ be its Lie algebra. To the Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{p}$ being the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form B of $\mathfrak{g}$, there corresponds the dual $\mathfrak{u}=\mathfrak{k}+(-1)^{1/2}\mathfrak{p}$. Denote by U the connected compact group whose Lie algebra is $\mathfrak{u}$ and by T a maximal torus of U. We denote by dG_0, dU, dK, dT, $d(G_0/K)$, $d(U/K)$, $d(U/T)$ the volume elements corresponding to the Riemannian structure on each space obtained naturally by B (cf. [4, Chapter IV]). We also denote by $\kappa(M)$ the Gaussian curvature of a Riemannian manifold M and by $\chi(M)$ the Euler-Poincaré characteristic. One has $\kappa(U/K)=(-1)^{m/2}\kappa(G_0/K)$ with $m=\dim(G_0/K)$. By the Gauss-Bonnet theorem (cf. [1], [2]), we have

$$(1) \qquad \frac{2\kappa(U/K)}{A_m}\int_{U/K} d(U/K) = \chi(U/K),$$

$$(2) \qquad \frac{2\kappa(U/T)}{A_{2N}}\int_{U/T} d(U/T) = \chi(U/T) = [W],$$

where A_m is the surface area of the unit m-sphere, N is the number of positive roots of $\mathfrak{g}$ and $[W]$ is the order of the Weyl group W of $\mathfrak{g}$. Eliminating the volume of U from (1), (2), one gets

$$(3) \qquad \int_K dK = \int_T dT\,\frac{A_{2N}[W]\kappa(U/K)}{A_m\chi(U/K)\kappa(U/T)}\,.$$

Let Γ be a discrete subgroup of G_0 such that