# STIELTJES INTEGRATION, SPECTRAL ANALYSIS, AND THE LOCALLY-CONVEX ALGEBRA (BV)

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The space (BV) (of all functions of bounded variation on an interval $[b_0, b_1]$) is an algebra under pointwise multiplication; one of our aims is to show how it can be made into a locally-convex algebra on which all continuous linear multiplicative functionals are represented by point measures on the interval $(b_0, b_1]$. Our main result deals with spectral and non spectral operators.

The algebra structure is disregarded in §5, where (BV) is endowed with a topology such that the most general continuous linear functional on (BV) has a natural representation by means of a Stieltjes integral.

Given a complete barreled space $\mathfrak{X}$, we introduce a family $\mathcal{E}_1$ of functions whose values are commuting projection operators in $\mathfrak{X}$. The algebra (BV) is topologized in such a way that each strongly-continuous representation $g \rightarrow u(g)$ (of (BV) on $\mathfrak{X}$) can be expressed in a natural way in terms of some $F \in \mathcal{E}_1$; in fact, $u(g)$ is the Stieltjes integral of $g$ with respect to $F$.

**1. A Helly theorem for Stieltjes integrals.** Let $\mathfrak{A}$ be an arbitrary complete locally-convex Hausdorff linear space; let $F$ be a bounded function on the interval $[b_0, b_1]$ into $\mathfrak{A}$, and let $g$ belong to the space (BV) of all complex-valued functions of bounded variation on $[b_0, b_1]$. Our basic theorem is as follows: if

(i) *the left-hand limit* $F(\alpha - 0)$ *exists whenever* $\alpha > b_0$,

(ii) $F(\beta) =$ *the right-hand limit* $F(\beta + 0)$ *whenever* $\beta < b_1$,

(iii) $F(b_1) = F(b_1 - 0)$ *and* $F(b_0) =$ *the zero-element of* $\mathfrak{A}$,

then the Stieltjes sums

$$\sum_{k=1}^{n} g(x_k)\{F(x_k) - F(x_{k-1})\}$$

(where $-\infty \leqq b_0 = x_0 < x_1 < \cdots < x_n = b_1 \leqq \infty$) converge to a limit, here denoted

$$\int g(\oplus[\lambda]) \cdot dF(\lambda) \tag{1}$$

—the limit is to be understood in the sense of refinements of subdivisions of $[b_0, b_1]$. If $g$ is left-continuous, then (1) coincides with the