VARIATIONAL METHODS FOR NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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In the present note, we give a simple general proof for the existence of solutions of the following two types of variational problems:

PROBLEM A. *To minimize* $\int_\Omega F(x, u, \cdots, D^m u)dx$ *over a subspace* V *of* $W^{m,p}(\Omega)$.

PROBLEM B. *To minimize* $\int_\Omega F(x, u, \cdots, D^m u)dx$ *for* u *in* V *with* $\int_\Omega G(x, u, \cdots, D^{m-1}u)dx = c$.

The solution of the first problem yields a weak solution of a corresponding elliptic boundary-value problem for the Euler-Lagrange equation

$$(1) \qquad Au = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha F_{p\alpha}(x, u, \cdots, D^m u) = 0.$$

From the solution of the second problem, we obtain a solution under corresponding boundary conditions of the nonlinear eigenvalue problem.

$$(2) \qquad Au = \lambda \left\{ \sum_{|\beta|\leq m-1} (-1)^{|\beta|} G_{p\beta}(x, u, \cdots, D^{m-1}u) \right\} = \lambda Bu, \quad \lambda \in R^1.$$

In §1, we give a complete self-contained treatment of the existence of minima of functionals on reflexive Banach spaces, a treatment which extends and strengthens earlier studies by Lusternik, E. Rothe, Vainberg, and others (see [6], [11], [12], [14], [15]). In §2, we apply the results of §1 to Problems A and B, above. In the case of Problem A, we strengthen and simplify results of Morrey [10] and Smale [13]. The relation of the resulting existence theorem for the solution of the variational boundary-value problem for equation (1) to those obtained by the writer in [2], [3], [4] by operator methods (as well as unpublished results of Leray and Lions) and the results of Višik [16] using other analytical methods, is discussed in detail in [5]. Special cases of the eigenvalue problem treated in Problem B have been treated for A linear by Levinson [7] with $A = \Delta$ on R^2, and by Berger [1] for general linear A.

1. Abstract variational problems. Let V be a real Banach space. Strong convergence in V is denoted by $\rightarrow$, weak convergence by $\rightharpoonup$. We consider two functions