AN EIGENVALUE PROBLEM FOR QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Eigenvalue problems for nonlinear equations have long been studied in the contexts of abstract function spaces and second-order ordinary differential equations. The present note treats such problems for certain quasi-linear elliptic partial differential equations by means of functional analysis on Sobolev spaces, and extends work in this direction by Levinson [7], Golomb [6], Duff [5], and Vaĭnberg [8]. The variational method used is a direct generalization of the linear case and thus allows the introduction of a simple Hilbert-space approach to this problem.

1. Let G be a fixed bounded domain in real Euclidean N-space R^N with boundary G and closure $\bar{G}=G\cup\partial G$. A general point of G will be denoted $x=(x_1, x_2, \cdots, x_n)$. Integration over G will always be taken with respect to Lebesgue N-dimensional measure. All derivatives are taken in the generalized sense of L. Schwartz. The following notation is very convenient: the elementary differential operators are written

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \qquad (1 \leqq j \leqq N),$$

and for any N-tuple of non-negative integers $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_N)$ the corresponding differential operator of order $|\alpha| = \alpha_1+\alpha_2+ \cdots +\alpha_N$ is written $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N}$. A linear operator A of order $2m$ is said to be in divergence form if it can be written:

$$Au = \sum_{|\alpha|,|\beta|\leqq m} D^\alpha(a_{\alpha\beta}(x)D^\beta u).$$

If $a_{\alpha\beta}(x)=a_{\beta\alpha}(x)$, A is also formally self-adjoint.

A real linear differential operator A is uniformly elliptic in G if the

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