WHITEHEAD GROUPS OF FREE ASSOCIATIVE ALGEBRAS

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Let R be a principal ideal domain, X a set, and Λ the free associative algebra over R on the set X. Then Λ is a supplemented algebra over R, where the augmentation $\epsilon_\Lambda: \Lambda \to R$ is the unique map of algebras extending $x \to 0$, $x \in X$, given by the universal property of Λ. We denote $\overline{K}_1(\Lambda) = \text{coker } \eta_{\Lambda *}: K_1(R) \to K_1(\Lambda)$, where $\eta: R \to \Lambda$ is the unit.[1]

THEOREM 1. $\overline{K}_1(\Lambda) = 0$, *or, equivalently,* $\eta_{\Lambda *}: K_1(R) \to K_1(\Lambda)$ *is an isomorphism.*

We remark that Theorem 1 applies to the case $R = Z$, the ring of integers, or R = any field. Since $\eta_{\Lambda *}$ is a monomorphism for functorial reasons ($\epsilon_\Lambda \eta_\Lambda = 1: R \to R$), the two assertions of Theorem 1 are seen to be equivalent.

LEMMA 1. *Any regular matrix T over Λ is equivalent by elementary operations to a regular matrix of the form*

$$M = M_0 + M_1 x_1 + M_2 x_2 + \cdots + M_n x_n,$$

where M_i $(0 \leqq i \leqq n)$ are matrices over R and $x_1, x_2, \cdots, x_n$ are distinct elements of X.

The proof is a standard exercise and will be omitted (see also [3]).

Using the notation of Lemma 1, if we apply ϵ_Λ, we see that M_0 is a regular matrix over R. Thus, $[M] = [M_0^{-1} M] \in \overline{K}_1(\Lambda)$, and $[M] \in \overline{K}_1(\Lambda)$ is represented by an $m \times m$ matrix of the form

$$N = 1 + N_1 x_1 + N_2 x_2 + \cdots + N_n x_n, \tag{1}$$

where N_i $(1 \leqq i \leqq n)$ are matrices over R, and $x_1, x_2, \cdots, x_n$ are distinct elements of X.

LEMMA 2. *The subalgebra (without unit) $\mathfrak{N}$, generated by $N_1, N_2, \cdots, N_n$, of the ring of endomorphisms $E(R, m)$ of a free R-module of rank m, is nilpotent.*

PROOF. Since N is regular, there is a matrix

[1] If R is a ring (associative, with unit), then $K_1(R) = \text{GL}(R)/\mathcal{E}(R)$ where $\text{GL}(R) = \text{dir}_\to\text{limit GL}(n, R)$ and $\mathcal{E}(R) = \text{dir}_\to\text{limit } \mathcal{E}(n, R)$, where $\mathcal{E}(n, R)$ is the subgroup of $\text{GL}(n, R)$ generated by elementary matrices (see Bass [1]).