A RADON-NIKODYM THEOREM IN W^*-ALGEBRAS[1]

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1. **Introduction.** The purpose of this paper is to show a Radon-Nikodym theorem in general W^*-algebras as follows: Let M be a W^*-algebra, and ϕ, ψ two normal positive linear functionals on M such that $\psi \leqq \phi$; then there is a positive element t_0 of M with $0 \leqq t_0 \leqq 1$ satisfying $\psi(x) = \phi(t_0 x t_0)$ for all $x \in M$ (Theorem 2). This theorem is the affirmative solution to a problem raised by Dixmier [1, p. 63] and the author [3, p. 1.46 and Question 2 in the appendix]. A less cogent Radon-Nikodym theorem in general W^*-algebras has been proved by the author [3, p. 1.46].

2. **Theorems.** To prove the above theorem, we shall provide some considerations.

Let M be a W^*-algebra, ϕ a normal positive linear functional on M. For $a, x \in M$, put $(Ra\phi)(x) = \phi(xa)$; then $Ra\phi$ is a σ-continuous linear functional on M. Then we shall show

PROPOSITION 1. *Suppose that $Ra\phi$ is self-adjoint; then we have* $|(Ra\phi)(h)| = |\phi(ha)| \leqq \|a\|\phi(h)$ for $h\ (\geqq 0) \in M$.

PROOF. By the assumption, $(Ra\phi)^*(x) = [(Ra\phi)(x^*)]^- = [\phi(x^*a)]^- = [\phi((a^*x)^*)]^- = \phi(a^*x) = (Ra\phi)(x) = \phi(xa)$ for $x \in M$.

Hence $\phi(a^*x) = \phi(xa)$, so that $\phi(xa^2) = \phi(xaa) = \phi(a^*xa)$; therefore $Ra^2\phi \geqq 0$ and so, analogously, we have $\phi(xa^4) = \phi((a^2)^*xa^2)$.

By the analogous discussion, we have

$$\phi(xa^{2^{n+1}}) = \phi((a^{2^n})^*x(a^{2^n})) \quad \text{for } x \in M.$$

Then, for $h \geqq 0$,

$$\begin{aligned} |\phi(ha)| &= |\phi(h^{1/2}h^{1/2}a)| \leqq \phi(h)^{1/2}\phi(a^*ha)^{1/2} \\ &= \phi(h)^{1/2}\phi(ha^2)^{1/2} \leqq \phi(h)^{1/2}\{\phi(h)^{1/2}\phi((a^2)^*ha^2)^{1/2}\}^{1/2} \\ &= \phi(h)^{1/2}\phi(h)^{1/4}\phi(ha^4)^{1/4} = \phi(H)^{1/2+1/4}\phi(ha^4)^{1/4} \\ &= \cdots\cdots\cdots\cdots \\ &= \phi(h)^{\sum_{i=1}^{n}(1/2^i)}\,\phi(ha^{2^n})^{1/2^n} = \phi(h)^{1-1/2^n}\,\phi(ha^{2^n})^{1/2^n} \\ &\leqq \phi(h)^{1-1/2^n}(\|\phi\|\|h\|\|a\|^{2^n})^{1/2^n} \\ &= \phi(h)^{1-1/2^n}\|a\|(\|\phi\|\|h\|)^{1/2^n} \to \|a\|\phi(h) \qquad (n \to \infty). \end{aligned}$$

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