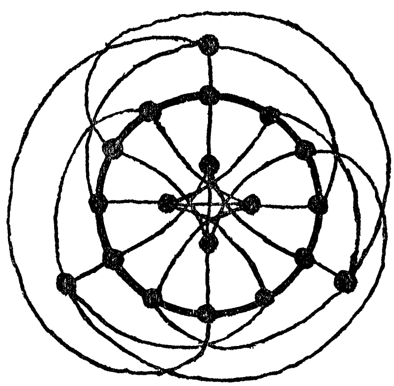
THE SMALLEST GRAPH OF GIRTH 5 AND VALENCY 4

BY NEIL ROBERTSON

Communicated by V. Klee, June 8, 1964

In what follows a graph S is constructed which has 19 vertices, valency 4 and girth 5. It is established that, to an isomorphism, S is the only graph with less than 20 vertices of valency 4 and girth 5. Some of its elementary properties are pointed out. This graph represents a continuation of the studies by Tutte [3], McGee [1] and Singleton [2], on graphs with specified valency and girth containing a relatively small number of vertices.



THE GRAPH S

Suppose G is a graph satisfying the stated constraints. The arcs of length 2 from any vertex x in G form a subtree $T(G, x)$ with 17 vertices. If G contains only these 17 vertices, count the pentagons through an edge A incident with x. Counting the arcs of length 2 proceeding away from the end of A opposite to x, it is clear that there are 9 pentagons through A. There must then be $9 \cdot 17 \cdot \frac{4}{5} \cdot 2$ pentagons in G, which is absurd. When G has 18 vertices each edge is contained in 8 pentagons, giving $8 \cdot 18 \cdot \frac{4}{5} \cdot 2$ pentagons in G, which is again impossible. G must thus contain at least 19 vertices. In this event let the 2 vertices outside $T(G, x)$ be nonadjacent for each vertex x in G. Then each edge is in 7 pentagons and G contains $7 \cdot 19 \cdot \frac{4}{5} \cdot 2$ pentagons. This contradiction implies that for some x in G the 2 extra vertices must be adjacent. There are only two essentially different ways in which their edges can be connected to $T(G, x)$. One of these configurations completes uniquely to the graph S while the other cannot be