NOETHERIAN SIMPLE RINGS

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THEOREM 1. *A right noetherian simple ring R with identity is isomorphic to the endomorphism ring of a unital torsion-free module M of finite rank over an integral domain.*

Since any right artinian ring with identity is right noetherian, this theorem generalizes the classical Wedderburn-Artin theorem which states that a right artinian simple ring with identity is (in the sense of isomorphism) the ring of endomorphisms of a unital module V over a (not necessarily commutative) field D.

The implications of this theorem for the structure of R are not yet apparent to the author. For instance, does it imply that R must contain a nontrivial idempotent, if R is not an integral domain?

The conclusion of Theorem 1 holds for any simple ring R with identity which satisfies the maximum conditions on annihilator right ideals and complement right ideals. According to Goldie [2] R will then have a classical right quotient ring R which is a simple artinian ring, that is, a full ring D_n of $n\times n$ matrices over a field D. Actually, we prove the theorem in the following setting.

THEOREM 2. *If R is a simple ring with identity which contains a minimal complement ($=$closed$=$uniform) right ideal, then R is the endomorphism ring of a unital torsion-free module over an integral domain.*

OUTLINE OF THE PROOF. By a theorem of Utumi [3], the maximal right quotient ring S of R is a full ring of l.t.'s in a right vector space over a field D. It is easily checked that there exists a primitive idempotent $e\in S$ such that $K=eSe\cap R\neq 0$. Then $D=eSe$ is a field, $V=Se$ is a right vector space over D, and S is naturally isomorphic to $\Omega=\mathrm{Hom}_D(V,\ V)$ under a map ϕ which assigns to each $s\in S$ the element $\phi(s)$ which satisfies $\phi(s)x=sx$, for all $x\in V$. Since D is the right quotient field of K (Faith and Utumi [1]), then D is the right quotient field of the subring Δ generated by K and e. Furthermore, $M=Se\cap R$ is a unital torsion-free module over Δ and it can be shown that $V=MD=\{xd^{-1}|\,x\in M,\ 0\neq d\in\Delta\}$. Therefore any element γ in $\Gamma=\mathrm{Hom}_\Delta(M,\ M)$ has a unique extension γ' in Ω. The natural isomorphism $S\cong\Omega$ implies that Γ is isomorphic to the subring $T=\{s\in S|\,sM\subseteq M\}$. Since T contains R, in order to establish