SETS OF UNIQUENESS FOR SOME CLASSES OF TRIGONOMETRICAL SERIES[1]

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The group of real numbers modulo 2π is denoted by T. The Fourier coefficients of a distribution μ on T are denoted, as usual, by $\hat{\mu}(n)$. If B is a space of (doubly infinite) sequences we denote by $\mathfrak{F}B$ the space of those distributions μ on T for which $\{\hat{\mu}(n)\}\in B$; and if B is normed we put $\|\mu\|_{\mathfrak{F}B}=\|\{\hat{\mu}(n)\}\|_B$. A closed set E is a set of uniqueness for B if E carries no element of $\mathfrak{F}B$. The purpose of this note is to construct sets of positive measure which are sets of uniqueness for l^p, $1\leqq p<2$.

LEMMA. *Let $\epsilon>0$, $1\leqq p<2$. There exists a closed set $E_{\epsilon,p}\subset T$ having the following properties:*

(1) measure $(E_{\epsilon,p})>2\pi-\epsilon$.

(2) *If μ is carried by $E_{\epsilon,p}$ then* $\|\mu\|_{\mathfrak{F}l^\infty}\leqq\epsilon\|\mu\|_{\mathfrak{F}l^p}$.

PROOF. Let $\gamma>0$. Put

$$f_\gamma(x)=\begin{cases}\dfrac{\gamma-2\pi}{\gamma} & (0<x<\gamma \mod 2\pi),\\ 1 & (\gamma\leqq x<2\pi \mod 2\pi).\end{cases}$$

Then, by the theorem of Hausdorff-Young (or by direct computation),

$$\|f_\gamma\|_{\mathfrak{F}l^q}\leqq K_p\|f_\gamma\|_{L^p}\sim K_p\gamma^{1/p-1},\quad \text{where}\quad 1/p+1/q=1.$$

Choose an integer N (large) and integers $\lambda_1,\cdots,\lambda_N$ "lacunary" enough so that, taking $\gamma=\epsilon/N$, we have

$$\left\|\frac{1}{N}\sum_1^N f_\gamma(\lambda_j x)\right\|_{\mathfrak{F}l^q}\leqq 2N^{1/q-1}\|f_\gamma\|_{\mathfrak{F}l^q}.$$

Denote

$$F(x)=\frac{1}{N}\sum_1^N f_\gamma(\lambda_j x);$$

then,

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