MINIMAL SETS AND ERGODIC MEASURES IN $\beta N - N$

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If $t(n) = n+1$ for $n \in N$ (N is the positive integers, βN its Stone-Čech compactification), then t extends uniquely to a continuous mapping (again called t) of βN into βN, and the restriction of t to N^* $= \beta N - N$ is a homeomorphism of N^* onto N^* [**R**, Theorem 4]. If $f \in C(\beta N)$ (the space of continuous real-valued functions on βN), let f_t be determined by $f_t(n) = f(n+1)(n \in N)$. Then for $\omega \in N^*$, $t(\omega)$ is characterized by the relation $f(t(\omega)) = f_t(\omega)(f \in C(\beta N))$.

THEOREM. *Every t-invariant, compact, nonempty set $S \subset N^*$, which is minimal with respect to these properties, is the support of at least two ergodic t-invariant Borel probability measures.*

To say that μ is ergodic is to say that every t-invariant Borel set A in N^* has $\mu(A) = 1$ or $\mu(A) = 0$. We notice that [**BH**, Theorem 2] in our context says:

LEMMA. *If W is the set of all t-invariant Borel probability measures on S, the extreme points of W are exactly the ergodic measures on S.*

PROOF OF THEOREM. Let S be a minimal set, let $\omega \in S$ and put $T_n f = (1/n) \sum_{i=1}^{n} f(t^i \omega)$ $(f \in C(\beta N); n = 1, 2, \cdots)$. Each T_n is a positive linear functional of norm 1. The set $\{T_n\}$ has at least two limit points in the weak-* topology of the dual space of $C(\beta N)$, for otherwise the sequence $T_n f$ would converge for every $f \in C(\beta N)$, denying Theorem 6 of [**R**].

Let L_1 and L_2 be limit points of $\{T_n\}$. By the Riesz representation theorem, there exist Borel measures μ_1 and μ_2 on βN such that $L_i f = \int f \, d\mu_i$ $(i = 1, 2; f \in C(\beta N))$; these are probability measures because each L_i is positive and of norm 1, and they are t-invariant because $L_i f_t = L_i f$ $(i = 1, 2; f \in C(\beta N))$. Furthermore, they are carried by S because $T_n f = 1$ for every $f \in C(\beta N)$ whose value at all points of S is 1, and for all n; hence $L_i f = 1$ for such f $(i = 1, 2)$.

Thus the set W of all t-invariant probability measures on S has at least two points. Since W is convex and weak-* compact, the Krein-Milman theorem implies that W has at least two extreme points. By the Lemma, each of these is ergodic. Finally, S is the support of

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