THE TEICHMÜLLER SPACE OF AN ARBITRARY FUCHSIAN GROUP[1]

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1. **Introduction.** Let U be the upper half plane. Let Σ be the set of quasiconformal self-mappings of U which leave 0, 1, and ∞ fixed. The universal Teichmüller space of Bers is the set T of mappings $h: R \rightarrow R$ which are boundary values of mappings in Σ.

Let M be the open unit ball in $L_\infty(U)$. For each μ in M, let f^μ be the unique mapping in Σ which satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z. \tag{1}$$

We map M onto T by sending μ to the boundary mapping of f^μ. T is given the quotient topology induced by the L_∞ topology on M. The right translations, of the form $h \rightarrow h \circ h_0$, are homeomorphisms of T.

We shall also associate to each μ in M a function ϕ^μ holomorphic in the lower half plane U^*. For each μ, let w^μ be the unique quasiconformal mapping of the plane on itself which is conformal in U^*, satisfies (1) in U, and leaves 0, 1, and ∞ fixed. ϕ^μ is the Schwarzian derivative $\{w^\mu, z\}$ of w^μ in U^*. By Nehari [3], ϕ^μ belongs to the Banach space B of holomorphic functions ψ on U^* which satisfy

$$\|\psi\| = \sup |(z - z^*)^2 \psi(z)| < \infty.$$

It is known [1, pp. 291–292] that $\phi^\mu = \phi^\nu$ if and only if f^μ and f^ν have the same boundary values. Hence, there is an injection $\theta: T \rightarrow B$ which sends the boundary function of f^μ to ϕ^μ. We shall write $\theta(T) = \Delta$.

Now let G be a Fuchsian group on U; that is, a discontinuous group of conformal self-mappings of U, not necessarily finitely generated. The mapping f in Σ is compatible with G if $f \circ A \circ f^{-1}$ is conformal for every A in G. The Teichmüller space $T(G)$ is the set of h in T which are boundary values of mappings compatible with G. The space $B(G)$ of quadratic differentials is the set of ϕ in B such that

$$\phi(Az)A'(z)^2 = \phi(z) \qquad \text{for all } A \text{ in } G.$$

Ahlfors proved in [1] that Δ is open in B. Bers [2] proved that θ maps T homeomorphically on Δ and maps $T(G)$ onto an open subset of $B(G)$. These results are summed up in the following theorems:

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