ON THE UNKNOTTEDNESS OF THE FIXED POINT SET OF DIFFERENTIABLE CIRCLE GROUP ACTIONS ON SPHERES—P. A. SMITH CONJECTURE

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The original P. A. Smith conjecture is that there are no Z_p actions on S^3 with a knotted S^1 as fixed point set. The so-called generalized P. A. Smith conjecture is that there are no Z_p or circle group actions on S^n with a knotted S^{n-2} as fixed point set [2], [8]. Mazur [5], [6] tried to give counterexamples for the cases $n=4$, 5 but there are several mistakes. In this paper, we show that the P. A. Smith conjecture is true for differentiable circle group actions. According to Giffen [3], there are examples of differentiable Z_p actions on S^n, $n \geqq 5$, p arbitrary, with knotted S^{n-2} as their fixed point sets.

In view of the fact that the cohomological theories for Z_p actions and circle group actions are always parallel, it becomes more interesting to find the *differences* between Z_p actions and circle group actions. We will show that the circle group actions are more regular, in a sense, than Z_p actions.

THEOREM I. *Suppose given a differentiable action of S^1 on S^n, $n \neq 4$, with its fixed point set $F = S^{n-2}$, then F is necessarily unknotted. If $n=4$, then $S^n - F$ has the homotopy type of a circle. Actually, except for the cases $n=4$, 5, the following stronger result is true.*

THEOREM I′. *A differentiable action of S^1 on S^n with an $(n-2)$-dimensional fixed point set F is orthogonal if and only if F is an $(n-2)$-sphere.*

The above theorems are just special cases of the following classification theorem. First, we give a construction.

Construction. Given a compact contractible manifold X of dimension $n-1$, $n \geqq 5$, we may have a circle group action on the smoothed $D^2 \times X$ simply by letting $g \cdot (y, x) = (g \cdot y, x)$.

By h-cobordism theorem, $D^2 \times X$ is a differentiable disc. If we restrict the action to the boundary of $D^2 \times X$, we get a circle group action on S^n with its orbit space diffeomorphic to X and its fixed point set, F, diffeomorphic to ∂X.

THEOREM II. *For $n \geqq 5$, every differentiable circle group action on S^n with* dim $F = n-2$ *is differentiably equivalent to one and only one of the examples constructed above.*