ON CARDINALITIES OF ULTRAPRODUCTS

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Introduction. In the theory of models, the ultraproduct (or prime reduced product) construction has been a very useful method of forming models with given properties (see, for instance, [2]). It is natural to ask what the cardinality of an ultraproduct is when we are given the cardinalities of the factors. In this paper we obtain some new results in that direction; however, the questions stated explicitly in [2, p. 208], are still open.

Let us first mention briefly some of the known results. Throughout this note we shall let D be a nonprincipal ultrafilter over a set I of infinite power λ. Additional notation is explained in §1 below.

1. $\alpha \leqq \alpha^I/D \leqq \alpha^\lambda$ [2, p. 205].

2. If D is not countably complete, then $\prod_{i \in I} \alpha_i/D$ is either finite or of power at least 2^ω [2, p. 208].

3. If D is uniform, then $\lambda^I/D > \lambda$; moreover, $(2^{(\lambda)})^I/D = 2^\lambda$, where $2^{(\lambda)} = \sum_{\beta<\lambda} 2^\beta$ [2, p. 206].

4. There exists a D such that if α is infinite, then $\alpha^I/D = \alpha^\lambda$ [2, p. 207], [1, p. 399], and [3, p. 838]. (Two more general versions for products of cardinals are given in [1].)

We shall prove the following results.

THEOREM A. (i) *If α is infinite and D is not countably complete, then*

$$\alpha^I/D = (\alpha^I/D)^\omega.$$

(ii) *For any α, γ, and D,*

$$(\alpha^\gamma)^I/D \geqq (\alpha^I/D)^\gamma.$$

(iii) *If D is uniform then*

$$(\alpha^{(\lambda)})^I/D = (\alpha^I/D)^\lambda = \alpha^\lambda$$

where $\alpha^{(\lambda)} = \sum_{\beta<\lambda} \alpha^\beta$.

We introduce the notion of a (β, γ)-regular ultrafilter in §1, and use it to prove Theorem A and some more general results in §2.

1. **Regular ultrafilters.** We shall adopt all of the set-theoretical notation introduced in [1], including the notions of an ultraproduct $\prod_{i \in I} \alpha_i/D$ and ultrapower α^I/D of the cardinals α_i, α. We denote the set of all functions on X into Y by ${}^XY$. We let $S(X)$ be the set of all