ON INEQUALITIES OF WEAK TYPE

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1. **Introduction.** E. M. Stein [3] has recently established a general principle for obtaining inequalities satisfied by sequences of operators. Suppose that for finite measure space X one has a sequence of continuous linear transformations $\{T_n\}$ on $L^p(X)$ into $L^p(X)$, where $1 \leqq p \leqq 2$, and that further

$$\lim_{n \to \infty} T_n f(x)$$

exists a.e. for every $f(x)$ in $L^p(X)$. Then, under the assumption that X is a compact group[1] and the $\{T_n\}$ translation invariant, Stein was able to predict the existence of a constant Ω_0 such that

$$m\left[\left\{x: \sup_{1 \leqq k < \infty} |T_k f(x)| \geqq A\right\}\right] \leqq \frac{\Omega_0}{A^p} \int_X |f(x)|^p dx \tag{1}$$

for every $f(x)$ in $L^p(X)$ and $A > 0$. Conclusions of this sort are called inequalities of weak type, or, where $T^* f(x) = \sup_n |T_n f(x)| = T^*(x, f)$, that the operator T^* is of weak type (p, p). Inequality (1) has often appeared with convergence a.e. in analysis, but a single result of this generality, predicting (1) as a consequence of convergence, had never before been obtained. Indeed, it seems to contain most of the known examples of (1) in Fourier analysis as special cases. However, the general idea, that convergence alone might yield (1), seems to crop up in situations much removed from group theoretical considerations. For example, such inequalities are often the first step in proving theorems in probability and ergodic theory; it would be interesting to know to what extent they are implicit in the statement of their results as well. Using the general techniques of Stein and of ergodic theory, we have succeeded in proving an extension of this principle which includes many of these other results, as well as in throwing new light on old ones. Our results will only be stated here; complete details, together with other theorems of a similar nature, will appear elsewhere.

2. **Definitions and preliminaries.** Assume $w(x)$ is a measure-preserving transformation of a finite measure space $(X, \mathcal{L}, m)$, and let

[1] Or a homogeneous space of a compact group, as in rotations on the unit sphere in R_n.