THE CLOSING LEMMA AND STRUCTURAL STABILITY

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Introduction. Consider a differentiable n-manifold M. Let $\mathfrak{X} = \mathfrak{X}(M)$ be the space of all C^1 tangent vector fields on M under a C^1 topology [1]. Each $X \in \mathfrak{X}$ induces a C^1-flow on M called the X-flow. Let d be a metric on M and let ϵ be positive. Two flows are homeomorphic if there is a homeomorphism h of M onto itself taking the trajectories of one flow onto those of the other; the two flows are ϵ-homeomorphic if h can be chosen so that $d(h(p), p) < \epsilon$ for all $p \in M$. X is said to be structurally stable if, given $\epsilon > 0$, there then exists a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}$ such that for each $Y \in \mathfrak{U}$ the Y-flow is ϵ-homeomorphic to the X-flow. Let us say that X is crudely structurally stable if we drop the ϵ condition: X is crudely structurally stable if there exists a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}$ such that $Y \in \mathfrak{U}$ implies that the Y-flow is homeomorphic to the X-flow. Let Σ denote those X in $\mathfrak{X}$ which are structurally stable and let Σ_c denote those X in $\mathfrak{X}$ which are crudely structurally stable, obviously $\Sigma \subset \Sigma_c$. The problem of structural stability theory is to characterize Σ and Σ_c and to study the topological relation of Σ and Σ_c to $\mathfrak{X}$.

The most comprehensive results in structural stability theory are due to M. Peixoto [2], [3], [4] who has shown, when M is a compact 2-manifold, that $\Sigma = \Sigma_c$, $\bar{\Sigma} = \mathfrak{X}$, and that the fields in Σ are characterized completely as the fields with "generic" induced flows.

Related to the problem of structural stability is the following conjecture:

CLOSING LEMMA. *If the X-flow has a nontrivial recurrent trajectory through some $p \in M$ and if $\mathfrak{U}$ is any neighborhood of X in $\mathfrak{X}$ then there exists $Y \in \mathfrak{U}$ such that the Y-flow has a closed orbit through p.*

(Recall that a trajectory is nontrivially recurrent if it is contained in its α- or in its ω-limit set without being a closed orbit or a stationary point.)

Results concerning the Closing Lemma. M. Peixoto [4] has proved the Closing Lemma in the case that M is the 2-torus and X has no

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