CROSS-SECTIONS OF SOLUTION FUNNELS

BY CHARLES C. PUGH[1]

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1. Consider R^{n+1} as (t, y)-space where t is real and $y=(y^1, \cdots, y^n) \in R^n$. Let $\mathfrak{F}^n$ denote the set of all continuous maps $f: R^{n+1} \to R^n$ having compact support. For any $p=(t_0, y_0) \in R^{n+1}$ and $f \in \mathfrak{F}^n$, an f-solution through p is any C^1 map $y: R \to R^n$ such that $y(t)$ is a solution of the initial value problem

$$\frac{dy(t)}{dt} = f(t, y(t)), \qquad y(t_0) = y_0.$$

The f-funnel through p, $F(p)$, is the union of all the curves $(t, y(t))$ in R^{n+1} such that $y(t)$ is an f-solution. E. Kamke [3] introduced the term *integraltrichter* in 1932. (When f is Lipschitz continuous, then of course $F(p)$ is just the unique f-solution curve through p, but if f is only C^0 then $F(p)$ may consist of many f-solution curves.) For any real number s, the *cross-section* of $F(p)$ at time s is the set $K_s(p) = \{y \in R^n : (s, y) \in F(p)\}$.

DEFINITION. A subset A of R^m is a *funnel-section* if for some $n \geqq m$ there exist $f \in \mathfrak{F}^n$ and $p \in R^{n+1}$ such that $i(A) = K_s(p)$ for some real s, where $i: R^m \to R^n$ is the usual injection of R^m onto the span of the first m coordinate axes of R^n.

2. A theorem of H. Kneser [5] asserts that any funnel-section is a continuum (i.e., a compact, connected set). There naturally arises, then, the question: what are necessary and sufficient conditions that a continuum in R^m be a funnel-section? We prove the six theorems below as partial answers to this question.

THEOREM 1. *There exists a continuum P which is not a funnel-section.*

THEOREM 2. *There exists a funnel-section S which is not arcwise connected.*

P is a bounded outward spiral in $\mathbf{C} = R^2$ together with its limit circle:

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