THE NOTION OF DIMENSION IN THE THEORY OF ALGEBRAIC DIFFERENTIAL EQUATIONS[1]

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Consider a system of algebraic differential equations

$$P(y_1, \cdots, y_n) = 0 \qquad (P \in \Sigma)$$

with coefficients in a differential field $\mathfrak{F}$ (ordinary or partial); here Σ is any subset of the differential polynomial algebra $\mathfrak{A} = \mathfrak{F}\{y_1, \cdots, y_n\}$ over $\mathfrak{F}$. Denote the set of all solutions of this system by $\mathfrak{Z}(\Sigma)$. We seek a measure of the size of $\mathfrak{Z}(\Sigma)$. The analogous question for systems of algebraic equations (i.e. for affine algebraic geometry) has a satisfactory answer in the notion of dimension.

In the classical literature, where $\mathfrak{F}$ consists of meromorphic functions on some region of complex m-space, the solution is said to depend on a certain number d of arbitrary functions of m variables; if $d=0$ then the solution is said to depend on a certain number of arbitrary functions of $m-1$ variables; and so on. Of course, except in certain special cases, what this means (how these numbers are defined) is not made precise, and general results are therefore wanting.

The Ritt theory (see [1]) contains the beginning of a general answer to the question (when $\mathfrak{F}$ is of characteristic 0). First Σ is replaced by the perfect differential ideal $\mathfrak{a}$ generated by Σ; this is harmless since $\mathfrak{Z}(\Sigma) = \mathfrak{Z}(\mathfrak{a})$. Then $\mathfrak{a}$ is expressed as the intersection of its components, $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$; since $\mathfrak{Z}(\mathfrak{a}) = \mathfrak{Z}(\mathfrak{p}_1) \cup \cdots \cup \mathfrak{Z}(\mathfrak{p}_r)$, the question is reduced to the case in which Σ is a prime differential ideal $\mathfrak{p}$ of $\mathfrak{A}$. Finally, one takes a generic zero $\eta = (\eta_1, \cdots, \eta_n)$ of $\mathfrak{p}$, and computes the differential transcendence degree $d(\mathfrak{p})$ of the differential field extension $\mathfrak{F}\langle\eta\rangle$ of $\mathfrak{F}$; $d(\mathfrak{p})$ is called the *differential dimension* of $\mathfrak{p}$, or of $\mathfrak{Z}(\mathfrak{p})$, and is the "correct" definition for what is classically called the number of arbitrary functions of m variables in the solution of the system $P=0$ $(P \in \mathfrak{p})$. Moreover, if $\mathfrak{p}'$ is another prime differential ideal of $\mathfrak{A}$ subject to the inclusion $\mathfrak{p} \subset \mathfrak{p}'$ (or, equivalently, to the inclusion $\mathfrak{Z}(\mathfrak{p}) \supset \mathfrak{Z}(\mathfrak{p}')$) then $d(\mathfrak{p}) \geqq d(\mathfrak{p}')$; however, when the inclusions are strict the inequality need not be so. This shows that $d(\mathfrak{p})$ is not a sufficiently fine measure of the size of $\mathfrak{Z}(\mathfrak{p})$.

In what follows we present another measure, which is sufficiently fine, and describe its relation to $d(\mathfrak{p})$ and some of its other properties; it is vaguely reminiscent of Hilbert's "characteristic function" for

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