CONTINUITY PROPERTIES OF MONOTONE NONLINEAR OPERATORS IN BANACH SPACES

BY FELIX E. BROWDER[1]

Communicated March 12, 1964

In a number of recent papers, the writer ([1]–[11]) and G. J. Minty ([13]–[16]) have studied nonlinear functional equations in Banach spaces involving monotone operators satisfying various mild continuity conditions (in particular demicontinuity, hemicontinuity, and vague continuity). In his note [12], T. Kato has studied the inter-relations of these continuity assumptions for monotone operators and shown in particular that every hemicontinuous locally bounded monotone operator G from a Banach space X to its dual X^* is always demicontinuous. The writer has independently obtained this and related results by a slightly different method in connection with his study of multi-valued monotone nonlinear mappings [9]. We present this argument below in §1.

1. Let X be a complex Banach space, X^* the space of bounded conjugate-linear functionals on X, (w, u) the pairing between w in X^* and u in X. Following the notation of [12], $\rightarrow$ denotes strong convergence in X or X^*, $\rightharpoonup$ weak* convergence in X^*.

Let G be a function with domain $D=D(G)\subset X$ and values in X^*. Then:

(1) G is said to be *monotone* if

$$\operatorname{Re}(Gu - Gv, u - v) \geqq 0$$

for all u, v in D.

(2) G is said to be *demicontinuous* if $u_n \rightarrow u$ in D implies $Gu_n \rightharpoonup Gu$.

(3) G is said to be *hemicontinuous* if $u \in D$, $v \in X$ and $u+t_n v\in D$ where $t_n>0$, $t_n\rightarrow 0$, together imply $G(u+t_n v)\rightharpoonup Gu$.

(4) G is said to be *vaguely continuous* if $u\in D$, $v\in X$ and $u+tv\in D$ for $0<t<t_0$ for some $t_0>0$ imply that there exists a sequence $\{t_n\}$ with $t_n>0$ for all n, $t_n\rightarrow 0$ as $n\rightarrow +\infty$ such that $G(u+t_n v)\rightharpoonup Gu$.

(5) G is said to be *D-maximal monotone* if for $u_0\in D$, $w_0\in X^*$, the inequality $\operatorname{Re}(w_0 - Gu, u_0 - u)\geqq 0$ for all u in D implies that $w_0 = G(u_0)$.

(6) G is said to be *locally bounded* if for any sequence $\{u_n\}$ in D

[1] This research was supported in part by National Science Foundation grants NSF-G19751 and NSF-GP2283, and in part by Army Research Office (Durham) grant ARO(D)-31-124-G455.