DOUBLE LOOPS AND TERNARY RINGS

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1. Let T be a ternary ring with the ternary operation $T(a, b, c)$ and the distinguished elements 0, 1 (see [4]). On T, two loop structures can be defined by means of the binary operations $a+b=T(a, 1, b)$ and $ab=T(a, b, 0)$. The resulting loops are called the additive and the multiplicative loop of T, respectively. Together with $a0=0a=0$ they define the structure of a double loop on T, which satisfies moreover the following condition:

(1) For every $a, b\in T$, $b\neq 1$, the equation $x+a=xb$ is uniquely solvable.

The question arises whether any double loop, satisfying the necessary condition (1), can be the double loop of a ternary ring. Hughes [3] has answered this question for countably infinite loops, but with another definition of addition on T.

The purpose of this note is to present the affirmative answer to the question in the infinite case: any infinite double loop satisfying (1) can be given canonically the structure of a ternary ring (Theorem 1 below). It will also be established (Theorems 2m and 2a) that any infinite loop can be the additive or the multiplicative loop of a double loop satisfying (1), and hence of a ternary ring (see [2, I 4]).

Important special cases of ternary rings are those in which the ternary operation can be expressed as a linear combination of the two binary ones: $T(a, b, c)=ab+c$. In this note, such rings will be called linear. The double loop of a linear ternary ring satisfies the following conditions:

(2) For every $b, b', c, c'\in T$, $b\neq b'$, the equation $xb+c=xb'+c'$ is uniquely solvable.

(3) For every $a, a', d, d'\in T$, $a\neq a'$, the equations $ax+y=d$, $a'x+y=d'$ are solvable.

Conversely, any double loop satisfying (2), (3) can be made into a linear ternary ring by defining $T(a, b, c)=ab+c$.

The question, what loops can be additive loops of double loops satisfying (2), (3) has been answered by Hughes [3] for the case of countably infinite groups.

In this note, sufficient conditions will be given for an infinite loop (of any cardinality) to be the additive loop of a double loop satisfying (2), (3), and therefore of a linear ternary ring (Theorem 3). The conditions cover the case of infinite groups.